

# Chapter 10

## Electrodynamics

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We compute the electric fields due to static and moving charges, describe methods for computing the electric potential in boundary value problems, and solve Maxwell's equations numerically.

### 10.1 Static Charges

Suppose we want to know the electric field  $\mathbf{E}(\mathbf{r})$  at the point  $\mathbf{r}$  due to  $N$  point charges  $q_1, q_2, \dots, q_N$  at fixed positions  $r_1, r_2, \dots, r_N$ . We know that  $\mathbf{E}(\mathbf{r})$  satisfies a superposition principle and is given by

$$\mathbf{E}(\mathbf{r}) = K \sum_i^N \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i), \quad (10.1)$$

where  $\mathbf{r}_i$  is the fixed location of the  $i$ th charge and  $K$  is a constant that depends on the choice of units. One of the difficulties associated with electrodynamics is the competing systems of units. In the SI (or rationalized MKS) system of units, the charge is measured in coulombs (C) and the constant  $K$  is given by

$$K = \frac{1}{4\pi\epsilon_0} \approx 9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2. \quad (\text{SI units}) \quad (10.2)$$

The constant  $\epsilon_0$  is known as the electrical permittivity of free space. This choice of units is not convenient for computer programs because  $K \gg 1$ . Another popular system of units is the Gaussian (cgs) system for which the constant  $K$  is absorbed into the unit of charge so that  $K = 1$ . Charge is in “electrostatic units” or esu. One feature of Gaussian units is that the electric and magnetic fields have the same units. For example, the (Lorentz) force on a particle of charge  $q$  and velocity  $\mathbf{v}$  in an electric field  $\mathbf{E}$  and a magnetic field  $\mathbf{B}$  has the form

$$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}). \quad (\text{Gaussian units}) \quad (10.3)$$

These virtues of the Gaussian system of units lead us to adopt this system for this chapter even though SI units are used in introductory texts.

The usual way of visualizing the electric field is to draw *electric field lines*. The properties of these lines are as follows:

1. An electric field line is a directed line whose tangent at every position is parallel to the electric field at that position.
2. The lines are smooth and continuous except at singularities such as point charges. (It makes no sense to talk about the electric field *at* a point charge.)
3. The density of lines at any point in space is proportional to the magnitude of the field at that point. This property implies that the total number of electric field lines from a point charge is proportional to the magnitude of that charge. The value of the proportionality constant is chosen to provide the clearest pictorial representation of the field. The drawing of field lines is art plus science.

**Program fieldline**, which is listed in the following, draws electric field lines in two dimensions starting at positive charges if the net charge  $q_{\text{net}} \geq 0$  or at negative charges if  $q_{\text{net}} < 0$ . The program implements the following algorithm:

1. Begin at a point  $(x, y)$  near a charge and compute the components  $E_x$  and  $E_y$  of the electric field vector  $\mathbf{E}$  using (9.1).
2. Draw a small line segment of size  $\Delta s = |\Delta \mathbf{s}|$  tangent to  $\mathbf{E}$  at that point. If  $q_{\text{net}} \geq 0$ , then  $\Delta s > 0$ , otherwise  $\Delta s < 0$ . The components of the line segment are given by

$$\Delta x = \Delta s \frac{E_x}{|\mathbf{E}|} \text{ and } \Delta y = \Delta s \frac{E_y}{|\mathbf{E}|}. \quad (10.4)$$

The program uses a small value for  $\Delta s$  if the field line is close to the charges or if the field magnitude is large. To speed up the program, a large value of  $\Delta s$  is used when a field line moves off the screen and the field has a small magnitude.

3. Repeat the process beginning at the new point  $(x + \Delta x, y + \Delta y)$ . Continue until the field line approaches another charge.
4. Repeat steps (1)–(3) for equally spaced starting positions on a circle around the charge. The spacing is inversely proportional to the magnitude of the charge.
5. Repeat steps (1)–(4) for each charge of the same sign.

**Program fieldline** draws the correct density of lines in most cases. In some cases a field line will go far away from the charges, but eventually return to a negative charge. Because such a field line might take too much time to draw, the user can hit any key to stop drawing this field line. There are some other situations where the algorithm breaks down. For example, if a field line is directed along the line connecting two charges that are equal in magnitude and opposite in sign, and begins by going away from both charges, then the field line will never return. Note the use of the **GET POINT** statement which allows the user to point to a position in the current window and click on the mouse to select it.

```

PROGRAM fieldline
! draw electric field lines in two dimensions
LIBRARY "csgraphics"
DIM x(10),y(10),q(10)
CALL screen(a,pos$,neg$)
CALL charges(N,x(),y(),q(),qtotal,a,pos$,neg$)
! draw field lines
CALL draw_lines(N,x(),y(),q(),qtotal,a)
END

SUB screen(a,pos$,neg$)
  LET L = 10
  CALL compute_aspect_ratio(L,xwin,ywin)
  SET WINDOW -xwin,xwin,-ywin,ywin
  LET a = 0.2           ! "radius" of visual image of charges
  SET COLOR "blue"
  BOX CIRCLE -a,a,-a,a
  FLOOD 0,0
  BOX KEEP -a,a,-a,a in pos$
  CLEAR
  SET COLOR "red"
  BOX CIRCLE -a,a,-a,a
  FLOOD 0,0
  BOX KEEP -a,a,-a,a in neg$
  CLEAR
END SUB

SUB charges(N,x(),y(),q(),qtotal,a,pos$,neg$)
! input charge values and location
LET N = 0           ! # of point charges
SET COLOR "black"
DO
  CALL draw_charges(N,x(),y(),q(),a,pos$,neg$)
  INPUT prompt "charge (0 to exit input mode) = ": charge
  IF charge <> 0 then
    PRINT "place mouse at charge location and click."
    LET N = N + 1
    GET POINT x(N),y(N)   ! location of charge
    LET q(N) = charge
    LET qtotal = qtotal + charge
  END IF
  CLEAR
LOOP until charge = 0
! redraw charges
CALL draw_charges(N,x(),y(),q(),a,pos$,neg$)
END SUB

```

```

SUB draw_charges(N,x(),y(),q(),a,pos$,neg$)
  FOR i = 1 to N
    IF q(i) > 0 then
      BOX SHOW pos$ at x(i)-a,y(i)-a
    ELSE
      BOX SHOW neg$ at x(i)-a,y(i)-a
    END IF
  NEXT i
END SUB

SUB draw_lines(N,x(),y(),q(),qtotal,a)
  ! number of lines per unit charge leaving positive charges
  SET COLOR "black"
  LET lpc = 8           ! lines per charge
  LET Emin = 0.01      ! if E < Emin stop drawing fieldline
  LET sign = sgn(qtotal)
  IF sign = 0 then LET sign = 1
  LET ds_small = 0.01*sign
  LET ds_big = sign
  FOR i = 1 to N       ! loop over all charges
    IF q(i)*sign > 0 then ! start fields at positive charges
      LET dtheta = 2*pi/(lpc*abs(q(i)))
      FOR theta = 0 to 2*pi step dtheta
        LET stop_plot$ = "no"
        LET xline = x(i) + a*cos(theta)
        LET yline = y(i) + a*sin(theta)
        DO
          LET Ex = 0
          LET Ey = 0
          FOR j = 1 to N
            ! x-distance from point to charge j
            LET dx = xline - x(j)
            ! y-distance from point to charge j
            LET dy = yline - y(j)
            LET r = sqr(dx*dx + dy*dy)
            IF r > 0.9*a then
              LET E0 = q(j)/(r*r*r)
              LET Ex = Ex + E0*dx
              LET Ey = Ey + E0*dy
            ELSE
              ! field line reached another charge
              LET stop_plot$ = "yes"
            END IF
          NEXT j
          LET E = sqr(Ex*Ex + Ey*Ey)
        LOOP UNTIL stop_plot$ = "yes"
      NEXT theta
    END IF
  NEXT i

```

```

        IF E > Emin or r < 20 then
            ! new position on fieldline
            LET xline = xline + ds_small*Ex/E
            LET yline = yline + ds_small*Ey/E
        ELSE
            ! new position on fieldline
            LET xline = xline + ds_big*Ex/E
            LET yline = yline + ds_big*Ey/E
        END IF
        PLOT xline,yline;
        IF key input then
            ! user can stop drawing field line
            GET KEY key
            LET stop_plot$ = "yes"
        END IF
        LOOP until stop_plot$ = "yes"
        PLOT          ! turn beam off
    NEXT theta
END IF
NEXT i
END SUB

```

*Problem 10.1.* Electric field lines from point charges

1. Program `fieldline` is written so that the user inputs the value of the charge from the keyboard and its position using the mouse. Enter some simple charge configurations and check that the program is working properly. Then let  $q(1) = 1$ ,  $q(2) = -4$ , and  $q(3) = 3$ , and place the three charges at the vertices of an approximate equilateral triangle. Are the units of charge and distance relevant? Remember that the number of field lines entering a negative charge might not be correct. Verify that the field lines never connect charges of the same sign. Why do field lines never cross?
2. Modify `SUB charge` so that values of  $q(i)$ ,  $x(i)$ , and  $y(i)$  can be assigned without entering them from the keyboard. In this way we can assign their values more precisely. Draw the field lines for an electric dipole.
3. Draw the field lines for the electric quadrupole with  $q(1) = 1$ ,  $x(1) = 1$ ,  $y(1) = 1$ ,  $q(2) = -1$ ,  $x(2) = -1$ ,  $y(2) = 1$ ,  $q(3) = 1$ ,  $x(3) = -1$ ,  $y(3) = -1$ , and  $q(4) = -1$ ,  $x(4) = 1$ , and  $y(4) = -1$ .
4. A continuous charge distribution can be approximated by a large number of closely spaced point charges. Draw the electric field lines due to a row of ten equally spaced unit charges located between  $-2.5$  and  $+2.5$  on the  $x$  axis. How does the electric field distribution compare to the distribution due to a single point charge?
5. Repeat part (d) with two rows of equally spaced positive charges on the lines  $y = 0$  and  $y = 1$ , respectively. Then consider one row of positive charges and one row of negative charges.

- Another way of representing the electric field is to divide space into a discrete grid and to draw arrows in the direction of  $\mathbf{E}$  at the vertices of the grid. The magnitude of the arrow can be chosen to be proportional to the magnitude of the electric field. Another possibility is to use color or gray scale to represent the magnitude. Which representation do you think conveys more information?

*Problem 10.2.* Field lines due to infinite line of charge

- `Program fieldline` plots field lines in two dimensions. Sometimes this restriction can lead to spurious results (see Freeman). Consider four identical charges placed at the corners of a square. Use `Program fieldline` to plot the field lines. What is wrong with the results? What should happen to the field lines near the center of the square?
- The two-dimensional analog of a point charge is an infinite line of charge perpendicular to the plane. The electric field due to an infinite line of charge is proportional to the linear charge density and inversely proportional to the distance (instead of the distance squared) from the line of charge to a point in the plane. Modify the calculation of the electric field in `Program fieldline` so that field lines from infinite lines of charge are drawn. Use your program to draw the field lines due to four identical infinite lines of charge located at the corners of a square, and compare the field lines with your results in part (a).
- Use your modified program from part (b) to draw the field lines for the two-dimensional analogs of the distributions considered in Problem 9.1. Compare the results for two and three dimensions, and discuss any qualitative differences.

*Problem 10.3.* Motion of a charged particle in an electric field

- Write a program to compute the motion of a particle of mass  $m$  and charge  $q$  in the presence of the electric field created by a fixed distribution of point charges. Use the Euler-Richardson algorithm to update the position and velocity of the particle. The acceleration of the charge is given by  $q\mathbf{E}/m$ , where  $\mathbf{E}$  is the electric field due to the fixed point charges. (We ignore the effects of radiation due to accelerating charges.) Incorporate the relevant subroutines from `Program fieldline` to visualize the electric field.
- Assume that  $\mathbf{E}$  is due to a charge  $q(1) = 1.5$  fixed at the origin. Simulate the motion of a charged particle of mass  $m = 0.1$  and charge  $q = 1$  initially at  $x = 1, y = 0$ . Consider the following initial conditions for its velocity: (i)  $v_x = 0, v_y = 0$ ; (ii)  $v_x = 1, v_y = 0$ ; (iii)  $v_x = 0, v_y = 1$ ; and (iv)  $v_x = -1, v_y = 0$ . Draw electric field lines beginning at the initial values of  $(x, y)$  and a line representing the particle's trajectory. Why does the trajectory of the particle not always follow a field line?
- Assume that the electric field is due to two fixed point charges:  $q(1) = 1$  at  $\mathbf{x}(1) = 2, \mathbf{y}(1) = 0$  and  $q(2) = -1$  at  $\mathbf{x}(2) = -2, \mathbf{y}(2) = 0$ . Place a charged particle of unit mass and unit positive charge at the point  $x = 0.05, y = 0$ . What do you expect the motion of this charge to be? Do the simulation and determine the qualitative nature of the motion.
- Consider the motion of a charged particle in the vicinity of the electric dipole defined in part (c). Choose the initial position to be five times the separation of the charges in the dipole. Do you find any bound orbits? Can you find any closed orbits or do all orbits show some precession?

We know that it is often easier to analyze the behavior of a system using energy rather than force concepts. We define the electric potential  $V(\mathbf{r})$  by the relation

$$V(\mathbf{r}_2) - V(\mathbf{r}_1) = - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E} \cdot d\mathbf{r} \quad (10.5)$$

or

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}). \quad (10.6)$$

Only differences in the potential between two points have physical significance. The gradient operator  $\nabla$  is given in Cartesian coordinates by

$$\nabla = \frac{\partial}{\partial x} \hat{\mathbf{x}} + \frac{\partial}{\partial y} \hat{\mathbf{y}} + \frac{\partial}{\partial z} \hat{\mathbf{z}}, \quad (10.7)$$

where the vectors  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$ , and  $\hat{\mathbf{z}}$  are unit vectors along the  $x$ ,  $y$ , and  $z$  axes respectively. If  $V$  depends only on the magnitude of  $\mathbf{r}$ , then (9.6) becomes  $E(r) = -dV(r)/dr$ . Recall that  $V(r)$  for a point charge  $q$  relative to a zero potential at infinity is given by

$$V(r) = \frac{q}{r}. \quad (\text{Gaussian units}) \quad (10.8)$$

The surface on which the electric potential has an equal value everywhere is called an *equipotential surface* (curve in two dimensions). Because  $\mathbf{E}$  is in the direction in which the electric potential decreases most rapidly, the electric field lines are orthogonal to the equipotential surfaces at any point. We can use this relation between the electric field lines and the equipotential lines to modify `Program fieldline` so that it draws the latter. Because the components of the line segment  $\Delta\mathbf{s}$  parallel to the electric field line are given by  $\Delta x = \Delta s(E_x/E)$  and  $\Delta y = \Delta s(E_y/E)$ , the components of the line segment perpendicular to  $\mathbf{E}$ , and hence parallel to the equipotential line, are given by  $\Delta x = -\Delta s(E_y/E)$  and  $\Delta y = \Delta s(E_x/E)$ . It is unimportant whether the minus sign is assigned to the  $x$  or  $y$  component, because the only difference would be the direction that the equipotential lines are drawn.

*Problem 10.4.* Equipotential lines

1. Modify `Program fieldline` to draw some of the equipotential lines for the charge distributions considered in Problem 9.1. One way to do so is to add a subroutine that uses a mouse click to determine the initial position of an equipotential line. The following code determines this position from a mouse click.

```
DO
  GET MOUSE x0,y0,s
LOOP until s <> 0
```

The variable `s` has the value 0 until the mouse is clicked. After a mouse click, the subroutine should draw the equipotential line starting from `x0,y0` and ending when the line closes on itself.

2. What would a higher density of equipotential lines mean if we drew lines such that each adjacent line differed from a neighboring one by a fixed potential difference?
3. Explain why equipotential surfaces never cross.

*Problem 10.5.* The electric potential due to a finite sheet of charge

Consider a uniformly charged dielectric plate of total charge  $Q$  and linear dimension  $L$  centered at  $(0, 0, 0)$  in the  $x$ - $y$  plane. In the limit  $L \rightarrow \infty$  with the charge density  $\sigma = Q/L^2$  a constant, we know that the electric field is normal to the sheet and is given by  $E_n = 2\pi\sigma$  (Gaussian units). What is the electric field due to a finite sheet of charge? A simple method is to divide the plate into a grid of  $p$  cells on a side such that each cell is small enough to be approximated by a point charge of magnitude  $q = Q/p^2$ . Because the potential is a scalar quantity, it is easier to compute the total potential rather than the total electric field from the  $N = p^2$  point charges. Use the relation (9.8) for the potential from a point charge and write a program to compute  $V(z)$  and hence  $E_z = -\partial V(z)/\partial z$  for points along the  $z$ -axis and perpendicular to the sheet. Take  $L = 1$ ,  $Q = 1$ , and  $p = 10$  for your initial calculations. Increase  $p$  until your results for  $V(z)$  do not change significantly. Plot  $V(z)$  and  $E_z$  as a function of  $z$  and compare their  $z$ -dependence to their infinite sheet counterparts.

\**Problem 10.6.* Electrostatic shielding

We know that the (static) electric field is zero inside a conductor, all excess charges reside on the surface of the conductor, and the surface charge density is greatest at the points of greatest curvature. Although these properties are plausible, it is instructive to do a simulation to see how these properties follow from Coulomb's law. For simplicity, consider the conductor to be two-dimensional so that the potential energy is proportional to  $\ln r$  rather than  $1/r$  (see Problem 9.2). It also is convenient to choose the surface of the conductor to be an ellipse.

1. If we are interested only in the final distribution of the charges and not in the dynamics of the system, we can use a Monte Carlo method. Our goal is to find the minimum energy configuration beginning with the  $N$  charges randomly placed within the ellipse. The method is to choose a charge  $i$  at random, and make a trial change in the position of the charge. The trial position should be no more than  $d_{\max}$  from the old position and still within the ellipse. The parameter  $d_{\max}$  should be chosen to be approximately  $b/10$ , where  $b$  is the semiminor axis of the ellipse. Compute the change in the total potential energy given by (in arbitrary units)

$$\Delta U = - \sum_j [\ln r_{ij}^{(\text{new})} - \ln r_{ij}^{(\text{old})}]. \quad (10.9)$$

The sum is over all charges in the system not including  $i$ . If  $\Delta U > 0$ , then reject the trial move, otherwise accept it. Repeat this procedure many times until very few trial moves are accepted. Write a program to implement this Monte Carlo algorithm. Run the simulation for  $N \geq 20$  charges inside a circle and then repeat the simulation for an ellipse. How are the charges distributed in the (approximately) minimum energy distribution? Which parts of the ellipse have a higher charge density?

2. Repeat part (a) for a two-dimensional conductor, but assume that the potential energy  $U \sim 1/r$ . Do the charges move to the surface? Is it sufficient that the interaction be repulsive?

3. Repeat part (a) with the added condition that there is a fixed positive charge of magnitude  $N/2$  located outside the ellipse. How does this fixed charge effect the charge distribution? Are the excess free charges still at the surface? Try different positions for the fixed charge.
4. Repeat parts (a) and (b) for  $N = 50$  charges located within an ellipsoid in three dimensions.

## 10.2 Numerical Solutions of Laplace's Equation

In Section ?? we found the electric fields and potentials due to a fixed distribution of charges. Suppose that we do not know the positions of the charges and instead know only the potential on a set of boundaries surrounding a charge-free region. This information is sufficient to determine the potential  $V(\mathbf{r})$  at any point within the charge-free region.

The direct method of solving for  $V(x, y, z)$  is based on Laplace's equation which can be expressed in Cartesian coordinates as

$$\nabla^2 V(x, y, z) \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad (10.10)$$

The problem is to find the function  $V(x, y, z)$  that satisfies (9.10) and the specified boundary conditions. This type of problem is an example of a *boundary value* problem. Because analytical methods for regions of arbitrary shape do not exist, the only general approach is to use numerical methods.

Laplace's equation is not a new law of physics, but can be derived directly from (9.6) and the relation  $\nabla \cdot \mathbf{E} = 0$  or indirectly from Coulomb's law in regions of space where there is no charge. For simplicity, we consider only two-dimensional boundary value problems for  $V(x, y)$ . We use a finite difference method and divide space into a discrete grid of points. In the following, we show that in the absence of a charge at  $(x, y)$ , the discrete form of Laplace's equation satisfies the relation

$$V(x, y) \approx \frac{1}{4} [V(x + \Delta x, y) + V(x - \Delta x, y) + V(x, y + \Delta y) + V(x, y - \Delta y)]. \quad (\text{two dimensions}) \quad (10.11)$$

That is,  $V(x, y)$  is the average of the potential at the four nearest neighbor points. This remarkable property of  $V(x, y)$  can be derived by approximating the partial derivatives in (9.10) by finite differences (see Problem 9.7b).

In Problem 9.7a we verify (9.12) by calculating the potential due to a point charge at a point in space selected by the user and at the four nearest neighbors. As the form of (9.12) implies, the average of the potential at the four neighboring points should equal the potential at the center point. We assume the form (9.8) for the potential  $V(r)$  due to a point charge, a form that satisfies Laplace's equation for  $r \neq 0$ . The program is listed in the following.

```
PROGRAM verify
! verify the discrete form of Laplace's equation
! for a point charge on a square lattice
LIBRARY "csgraphics"
```

```

CALL initial(L,dx,dy,#1,#2)
CALL draw_grid(L,dx,dy,#3)
DO
  CALL potential(L,dx,dy,stop_plot$,#1,#2,#3)
LOOP until stop_plot$ = "yes"
END

SUB initial(L,dx,dy,#1,#2)
  INPUT prompt "lattice dimension = ": L
  INPUT prompt "grid spacing in x direction = ": dx
  INPUT prompt "grid spacing in y direction = ": dy
  OPEN #1: screen 0,0.25,0.8,1
  SET WINDOW -1.2,2,-1,2
  OPEN #2: screen 0,0.25,0,0.6
END SUB

SUB draw_grid(L,dx,dy,#1)
  OPEN #1: screen 0.25,1,0,1
  CALL compute_aspect_ratio(L,xwin,ywin)
  SET WINDOW -xwin,xwin,-ywin,ywin
  LET a = 0.2*dx ! "radius" of visual image of charge
  SET COLOR "blue"
  BOX CIRCLE -a,a,-a,a
  FLOOD 0,0
  SET COLOR "black"
  FOR y = -L to L step dy
    FOR x = -L to L step dx
      PLOT POINTS: x,y
    NEXT x
  NEXT y
  BOX LINES -L,L,-L,L
END SUB

SUB potential(L,dx,dy,stop_plot$,#1,#2,#3)
  WINDOW #3
  SET CURSOR 1,20
  PRINT "click on mouse to choose site"
  LET s = 0
  DO
    GET MOUSE xs,ys,s
  LOOP until s = 2
  WINDOW #1
  CLEAR
  LET stop_plot$ = ""
  IF abs(xs) <= L and abs(ys) <= L then
    CALL showpotential(xs,ys,0,0,V0)
  
```

```

CALL showpotential(xs+dx,ys,1,0,V1)
CALL showpotential(xs-dx,ys,-1,0,V2)
CALL showpotential(xs,ys+dy,0,1,V3)
CALL showpotential(xs,ys-dy,0,-1,V4)
WINDOW #2
SET CURSOR 1,1
PRINT " average potential"
PRINT " of four neighbors:"
PRINT truncate(0.25*(V1+V2+V3+V4),4)
ELSE
  LET stop_plot$ = "yes"
END IF
END SUB

SUB showpotential(x,y,xp,yp,V)
  LET V = 1/sqr(x*x + y*y)      ! potential of a point charge
  LET V = truncate(V,4)
  PLOT TEXT, AT xp,yp: str$(V)
END SUB

```

*Problem 10.7.* Verification of the difference equation for the potential

1. Choose reasonable values for the grid spacings  $\Delta x$  and  $\Delta y$  and consider a point that is not too close to the source charge. Compare the computed potential at a point to the average of the potential at its four nearest neighbor points. Do similar measurements for other points. Does the relative agreement with (9.12) depend on the distance of the point to the source charge? Choose smaller values of  $\Delta x$  and  $\Delta y$  and determine if results are in better agreement with (9.12). Does it matter whether  $\Delta x$  and  $\Delta y$  have the same value?
2. Derive the finite difference equation (9.12) for  $V(x, y)$  using the second-order Taylor expansion:

$$V(x + \Delta x, y) = V(x, y) + \Delta x \frac{\partial V(x, y)}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 V(x, y)}{\partial x^2} + \dots \quad (10.12)$$

$$V(x, y + \Delta y) = V(x, y) + \Delta y \frac{\partial V(x, y)}{\partial y} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 V(x, y)}{\partial y^2} + \dots \quad (10.13)$$

The effect of including higher derivatives is discussed by MacDonald (see references).

Now that we have found that (9.12), a finite difference form of Laplace's equation, is consistent with Coulomb's law, we adopt (9.12) as the basis for computing the potential for systems for which we cannot calculate the potential directly. In particular, we consider problems where the potential is specified on a closed surface that divides space into interior and exterior regions in which the potential is independently determined. For simplicity, we consider only two-dimensional geometries. The approach, known as the *relaxation method*, is based on the following algorithm:

1. Divide the region of interest into a rectangular grid of points spanning the region. The region is enclosed by a surface (curve in two dimensions) with specified values of the potential along the curve.
2. Assign to a boundary point the potential of the boundary nearest the point.
3. Assign all interior points an arbitrary potential (preferably a reasonable guess).
4. Compute new values for the potential  $V$  for each interior point. Each new value is obtained by finding the average of the previous values of the potential at the four nearest neighbor points.
5. Repeat step (4) using the values of  $V$  obtained in the previous iteration. This iterative process is continued until the potential at each interior point is computed to the desired accuracy.

In Problems 9.8–9.10 we use this method and its variants to compute the potential for various geometries.

```

PROGRAM laplace
! implementation of Jacobi relaxation method to solve
! Laplace's equation in rectangular geometry
DIM V(0:100,0:100)
CALL assign(V(,),nx,ny,min_change)
CALL iterate(V(,),nx,ny,min_change,iterations)
END

SUB assign(V(,),nx,ny,min_change)
! linear dimension of rectangular region
LET nx = 9      ! number of interior points in x-direction
LET ny = 9      ! number of interior points in y-direction
LET V0 = 10     ! boundary potential of rectangle
INPUT prompt "percentage change = ": min_change
LET min_change = min_change/100
! fix potential on boundary of rectangle
FOR x = 0 to nx + 1
    LET V(x,0) = V0
    LET V(x,ny+1) = V0
NEXT x
FOR y = 0 to ny + 1
    LET V(0,y) = V0
    LET V(nx+1,y) = V0
NEXT y
! guess initial values of potentials of interior sites
FOR y = 1 to ny
    FOR x = 1 to nx
        LET V(x,y) = 0.9*V0
    
```

```

        NEXT x
    NEXT y
    CALL show_output(V(,),nx,ny,0)
END SUB

SUB iterate(V(,),nx,ny,min_change,iterations)
    DIM Vave(100,100)
    LET iterations = 0
    DO
        ! maximum difference of iterated potential at any site
        LET change = 0
        FOR y = 1 to ny
            FOR x = 1 to nx
                ! average of potential of neighboring cells
                LET Vave(x,y) = V(x+1,y) + V(x-1,y)
                LET Vave(x,y) = Vave(x,y) + V(x,y+1) + V(x,y-1)
                LET Vave(x,y) = 0.25*Vave(x,y)
                ! compute percentage change in potential
                IF Vave(x,y) <> 0 then
                    LET dV = abs((V(x,y) - Vave(x,y))/Vave(x,y))
                    IF dV > change then LET change = dV
                END IF
            NEXT x
        NEXT y
        FOR y = 1 to ny      ! update potential at each site
            FOR x = 1 to nx
                LET V(x,y) = Vave(x,y)
            NEXT x
        NEXT y
        LET iterations = iterations + 1
        CALL show_output(V(,),nx,ny,iterations)
    LOOP until change <= min_change
END SUB

SUB show_output(V(,),nx,ny,iterations)      ! print potential
    PRINT
    PRINT "iteration ="; iterations
    FOR y = 0 to ny + 1
        FOR x = nx + 1 to 0 step - 1
            PRINT using "###.##": V(x,y);
        NEXT x
        PRINT
    NEXT y
END SUB

```

*Problem 10.8.* Numerical solution of the potential within a rectangular region

1. Use Program `laplace` to determine the potential  $V(x, y)$  in a square region with linear dimension  $L = 9$ . The boundary of the square is at a potential  $V = 10$ . Let the number of interior grid points in the horizontal and vertical directions be  $n_x = n_y = 9$ , respectively. Before you run the program, guess the exact form of  $V(x, y)$  and set the initial values of the interior potential close to the exact answer. How many iterations are necessary to achieve 1% accuracy? Increase both  $\Delta x$  and  $\Delta y$  by a factor of two, and determine the number of iterations that are now necessary to achieve 1% accuracy.
2. Consider the same geometry as in part (a), but set the initial potential at the interior points equal to zero except for the center point whose potential is set equal to four. Does the potential distribution evolve to the same values as in part (a)? What is the effect of a poor initial guess? Are the final results independent of your initial guess?
3. Modify SUB `assign` in Program `laplace` so that the value of the potential at the four sides is 5, 10, 5, and 10, respectively (see Figure 9.1). Sketch the equipotential surfaces. What happens if the potential is 10 on three sides and 0 on the fourth? Start with a reasonable guess for the initial values of the potential at the interior points and iterate until 1% accuracy is obtained.
4. Consider the same initial choice of the potential as in part (b) and focus your attention on the potential at the points near the center of the square. If there were four random walkers at the central point corresponding to an initial potential of four, how many walkers would there be at the nearest neighbor points after the first iteration? Follow the distribution of the “walkers” as a function of the number of iterations and verify that the nature of the relaxation of the potential to its correct distribution is closely related to *diffusion* (see Chapters 7 and ??). It would be helpful to increase the number of points in the grid and the initial value of the potential at the central point to see the nature of the relaxation more clearly.

In Problem 9.8, we implemented a simple version of the relaxation method known as the Jacobi method. In particular, the new potential of each point is found based on the values of the potentials at the neighboring points at the previous iteration. After the entire lattice was visited, the potential at each point was updated *simultaneously*. The difficulty with this relaxation method is that it converges very slowly. The use of more general relaxation methods is discussed in many texts (cf. Koonin and Meredith or Press et al.). In Problem 9.9 we consider a method known as Gauss-Seidel relaxation.

*Problem 10.9.* Gauss-Seidel relaxation

1. Modify the program that you used in Problem 9.8 so that the potential at each point is updated sequentially. That is, after the average potential of the nearest neighbor points of point  $i$  is computed, update the potential at  $i$  immediately. In this way the new potential of the next point is computed using the most recently computed values of its nearest neighbor potentials. Are your results better, worse, or about the same as the simple relaxation method?
2. Imagine coloring the alternate points of a grid red and black, so that the grid resembles a checkerboard. Modify the program so that all the red points are updated first, and then all the black points are updated. This ordering is repeated for each iteration. Do your results converge any more quickly than in part (a)?

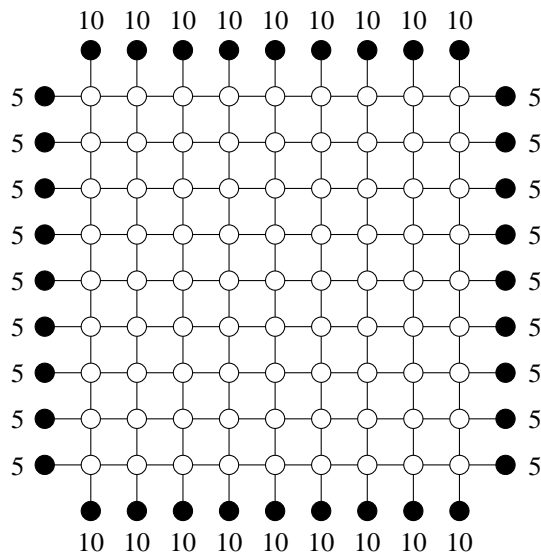


Figure 10.1: Potential distribution considered in Problem 9.8c. The number of interior points in each direction is nine.

3. The slow convergence of the relaxation methods we have explored is due to the fact that it takes a long time for a change in the potential at one point to effect changes further away. We can improve the Gauss-Seidel method by using an overrelaxation method which updates the new potential as follows:

$$\text{LET } V(x,y) = w * V_{\text{ave}}(x,y) + (1-w) * V(x,y)$$

The overrelaxation parameter  $w$  is in the range  $1 < w < 2$ . The effect of  $w$  is to cause the potential to change by a greater amount than in the simple relaxation procedure. Explore the dependence of the rate of convergence on  $w$ . A relaxation method that does increase the rate of convergence is explored in Project ??.

*Problem 10.10.* The capacitance of concentric squares

1. Use a relaxation method to compute the potential distribution between the two concentric square cylinders shown in Figure 9.2. The potential of the outer square conductor is  $V_{\text{out}} = 10$  and the potential of the inner square conductor is  $V_{\text{in}} = 5$ . The linear dimensions of the exterior and interior squares are  $L_{\text{out}} = 25$  and  $L_{\text{in}} = 5$ , respectively. Modify your program so that the potential of the interior square is fixed. Sketch the equipotential surfaces.
2. A system of two conductors with charge  $Q$  and  $-Q$  respectively has a capacitance  $C$  that is defined as the ratio of  $Q$  to the potential difference  $\Delta V$  between the two conductors. Determine the capacitance per unit length of the concentric cylinders considered in part (a). In this case  $\Delta V = 5$ . The charge  $Q$  can be determined from the fact that near a conducting

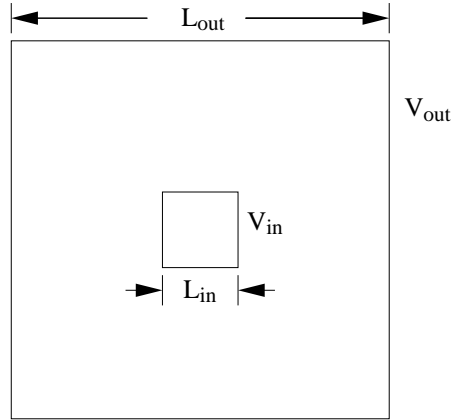


Figure 10.2: The geometry of the two concentric squares considered in Problem 9.10.

surface, the surface charge density  $\sigma$  is given by  $\sigma = E_n/4\pi$ , where  $E_n$  is the magnitude of the electric field normal to the surface.  $E_n$  can be approximated by the relation  $-\delta V/\delta r$ , where  $\delta V$  is the potential difference between a boundary point and an adjacent interior point a distance  $\delta r$  away. Use the result of part (a) to compute  $\delta V$  for each point adjacent to the two square surfaces. Use this information to determine  $E_n$  for the two surfaces and the charge per unit length on each conductor. Are the charges equal and opposite in sign? Compare your numerical result to the capacitance per unit length,  $1/2 \ln r_{\text{out}}/r_{\text{in}}$ , of a system of two concentric circular cylinders of radii  $r_{\text{out}}$  and  $r_{\text{in}}$ . Assume that the circumference of each cylinder equals the perimeter of the corresponding square, i.e.,  $2\pi r_{\text{out}} = 4L_{\text{out}}$  and  $2\pi r_{\text{in}} = 4L_{\text{in}}$ .

3. Move the inner square 1 cm off center and repeat the calculations of parts (a) and (b). How do the potential surfaces change? Is there any qualitative difference if we set the inner conductor potential equal to  $-5$  statvolt instead of  $+5$  statvolt?

Laplace's equation holds only in charge-free regions. If there is a charge density  $\rho(x, y, z)$  in the region, we need to use *Poisson's* equation which can be written as

$$\nabla^2 V(\mathbf{r}) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho(\mathbf{r}), \quad (10.14)$$

where  $\rho(\mathbf{r})$  is the charge density. The difference form of Poisson's equation is given in two dimensions by

$$\begin{aligned} V(x, y) \approx \frac{1}{4} [V(x + \Delta x, y) + V(x - \Delta x, y) + V(x, y + \Delta y) + V(x, y - \Delta y)] \\ + \frac{1}{4} \Delta x \Delta y 4\pi\rho(x, y). \end{aligned} \quad (10.15)$$

Note that the product  $\rho(x, y)\Delta x\Delta y$  is the total charge in the cell centered at  $(x, y)$ .

*Problem 10.11.* Numerical solution of Poisson's equation

1. Consider a square of linear dimension  $L = 25$  whose boundary is fixed at a potential equal to  $V = 10$ . Assume that each interior cell has a uniform charge density  $\rho$  such that the total charge is  $Q = 1$ . Use a modification of Program `laplace` to compute the potential distribution for this case. Compare the equipotential surfaces obtained for this case to that found in Problem 9.10.
2. Find the potential distribution if the charge distribution of part (a) is restricted to a  $5 \times 5$  square at the center.
3. Find the potential distribution if the charge distribution of part (a) is restricted to a  $1 \times 1$  square at the center. How does the potential compare to that of a point charge without the boundary?

### 10.3 Random Walk Solution of Laplace's Equation

In Section 9.2 we found that the solution to Laplace's equation in two dimensions at the point  $(x, y)$  is given by

$$V(x, y) = \frac{1}{4} \sum_{i=1}^4 V(i), \quad (10.16)$$

where  $V(i)$  is the value of the potential at the  $i$ th neighbor. A generalization of this result is that the potential at any point equals the average of the potential on a circle (or sphere in three dimensions) centered about that point.

The relation (9.18) can be given a probabilistic interpretation in terms of random walks (see Problem 9.8d). Suppose that many random walkers are at the point  $(x, y)$  and each walker "jumps" to one of its four neighbors (on a square grid) with equal probability  $p = 1/4$ . From (9.18) we see that the average potential found by the walkers after jumping one step is the potential at  $(x, y)$ . This relation generalizes to walkers that visit a point on a closed surface with fixed potential. The random walk algorithm for computing the solution to Laplace's equation can be stated as:

1. Begin at a point  $(x, y)$  where the value of the potential is desired, and take a step in a random direction.
2. Continue taking steps until the walker reaches the surface. Record  $V_b(i)$ , the potential at the boundary point  $i$ . A typical walk is shown in Figure 9.3.
3. Repeat steps (1) and (2)  $n$  times and sum the potential found at the surface each time.
4. The value of the potential at the point  $(x, y)$  is estimated by

$$V(x, y) = \frac{1}{n} \sum_{i=1}^n V_b(i) \quad (10.17)$$

where  $n$  is the total number of random walkers.

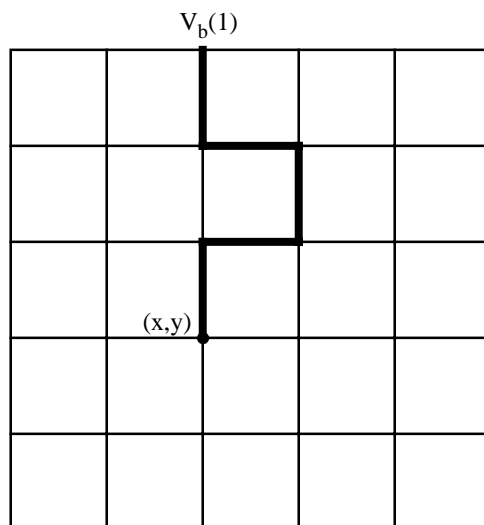


Figure 10.3: A random walk on a  $6 \times 6$  grid starting at the point  $(x, y) = (3, 3)$  and ending at the boundary point  $V_b(3, 6)$  where the potential is recorded.

*Problem 10.12.* Random walk solution of Laplace's equation

1. Consider the square region shown in Figure 9.1 and compare the results of the random walk method with the results of the relaxation method (see Problem 9.8c). Try  $n = 100$  and  $n = 1000$  walkers, and choose a point near the center of the square.
2. Repeat part (a) for other points within the square. Do you need more or less walkers when the potential near the surface is desired? How quickly do your answers converge as a function of  $n$ ?

The disadvantage of the random walk method is that it requires many walkers to obtain a good estimate of the potential at each point. However, if the potential is needed at only a small number of points, then the random walk method might be more appropriate than the relaxation method which requires the potential to be computed at all points within the region. Another case where the random walk method is appropriate is when the geometry of the boundary is fixed, but the potential in the interior for a variety of different boundary potentials is needed. In this case the quantity of interest is  $G(x, y, x_b, y_b)$ , the number of times that a walker from the point  $(x, y)$  lands at the boundary  $(x_b, y_b)$ . The random walk algorithm is equivalent to the relation

$$V(x, y) = \frac{1}{n} \sum_b G(x, y, x_b, y_b) V(x_b, y_b), \quad (10.18)$$

where the sum is over all points on the boundary. We can use the same function  $G$  for different distributions of the potential on a given boundary.  $G$  is an example of a Green's function, a function that you will encounter in advanced treatments of electrodynamics and quantum mechanics

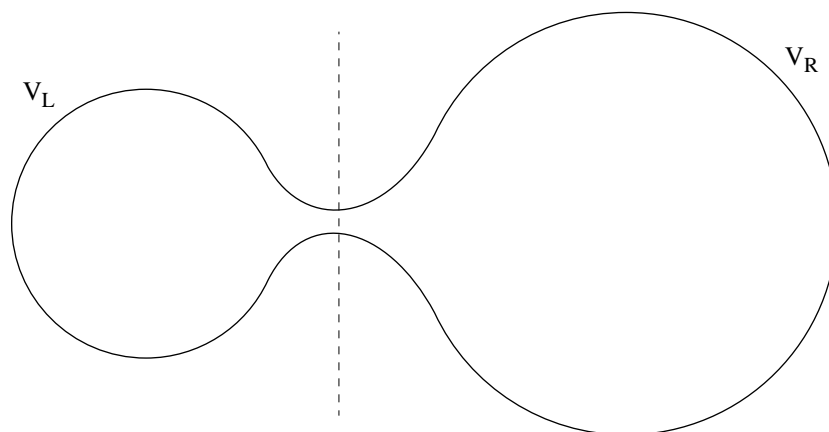


Figure 10.4: Two regions of space connected by a narrow neck. The boundary of the left region has a potential  $V_L$ , and the boundary of the right region has a potential  $V_R$ .

(cf. Section ??). Of course, if we change the geometry of the boundary, we have to recompute the function  $G$ .

*Problem 10.13.* Green's function solution of Laplace's equation

1. Compute the Green's function  $G(x, y, x_b, y_b)$  for the same geometry considered in Problem 9.12. Use at least 200 walkers at each interior point to estimate  $G$ . Because of the symmetry of the geometry, you can determine some of the values of  $G$  from other values without doing an additional calculation. Store your results for  $G$  in a file.
2. Use your results for  $G$  found in part (a) to determine the potential at each interior point when the boundary potential is the same as in part (a) except for five boundary points which are held at  $V = 20$ . Find the locations of the five boundary points that maximize the potential at the interior point located at  $(3, 5)$ . Repeat the calculation to maximize the potential at  $(5, 3)$ . Use trial and error guided by your physical intuition.

The random walk algorithm can help us gain additional insight into the nature of Laplace's equation. Suppose that you have a boundary similar to the one shown in Figure 9.4. The potentials on the left and right boundaries are  $V_L$  and  $V_R$ , respectively. If the neck between the two sides is narrow, it is clear that a random walker starting on the left side has a low probability of reaching the other side. Hence, we can conclude that the potential in the interior of the left side is approximately  $V_L$  except very near the neck.

Poisson's equation also can be solved using the random walk method. In this case, the potential is given by

$$V(x, y) = \frac{1}{n} \sum_{\alpha} V(\alpha) + \frac{\pi \Delta x \Delta y}{n} \sum_{i, \alpha} \rho(x_{i, \alpha}, y_{i, \alpha}), \quad (10.19)$$

where  $\alpha$  labels the walker, and  $i$  labels the point visited by the walker. That is, each time a walker is at a point  $i$ , we add the charge density at that point to the second sum in (9.21).

## 10.4 \*Fields Due to Moving Charges

The fact that accelerating charges radiate electromagnetic waves is one of the most important results in the history of physics. In this section we discuss a numerical algorithm for computing the electric and magnetic fields due to the motion of charged particles. The algorithm is very general, but requires some care in its application.

To understand the algorithm, we need a few results that can be found conveniently in Feynman's lectures. We begin with the fact that the scalar potential at the observation point  $\mathbf{R}$  due to a stationary particle of charge  $q$  is

$$V(\mathbf{R}) = \frac{q}{|\mathbf{R} - \mathbf{r}|}, \quad (10.20)$$

where  $\mathbf{r}$  is the position of the charged particle. The electric field is given by

$$\mathbf{E}(\mathbf{R}) = -\frac{\partial V(\mathbf{R})}{\partial \mathbf{R}}, \quad (10.21)$$

where  $\partial V(\mathbf{R})/\partial \mathbf{R}$  is the gradient with respect to the coordinates of the observation point. (Note that our notation for the observation point differs from that used in other sections in this chapter.) How do the relations (9.22) and (9.23) change when the particle is moving? We might guess that because it takes a finite time for the disturbance due to a charge to reach the point of observation, we should modify (9.22) by writing

$$V(\mathbf{R}) \stackrel{?}{=} \frac{q}{r_{\text{ret}}}, \quad (10.22)$$

where

$$r_{\text{ret}} = |\mathbf{R} - \mathbf{r}(t_{\text{ret}})|. \quad (10.23)$$

The quantity  $r_{\text{ret}}$  is the separation of the charged particle from the observation point  $\mathbf{R}$  at the retarded time  $t_{\text{ret}}$ . The latter is the time at which the particle was at  $\mathbf{r}(t_{\text{ret}})$  such that a disturbance starting at  $\mathbf{r}(t_{\text{ret}})$  and traveling at the speed of light would reach  $\mathbf{R}$  at time  $t$ ;  $t_{\text{ret}}$  is given by the implicit equation

$$t_{\text{ret}} = t - \frac{r_{\text{ret}}(t_{\text{ret}})}{c}, \quad (10.24)$$

where  $t$  is the observation time and  $c$  is the speed of light.

Although the above reasoning is plausible, the relation (9.24) is not quite correct (cf. Feynman et al. for a derivation of the correct result). We need to take into account that the potential due to the charge is a maximum if the particle is moving toward the observation point and a minimum if it is moving away. The correct result can be written as

$$V(\mathbf{R}, t) = \frac{q}{r_{\text{ret}}(1 - \hat{\mathbf{r}}_{\text{ret}} \cdot \mathbf{v}_{\text{ret}}/c)}, \quad (10.25)$$

where

$$\mathbf{v}_{\text{ret}} = \left. \frac{d\mathbf{r}(t)}{dt} \right|_{t=t_{\text{ret}}}. \quad (10.26)$$

To find the electric field of a moving charge, we recall that the electric field is related to the time rate of change of the magnetic flux. Hence, we expect that the total electric field at the observation point  $\mathbf{R}$  has a contribution due to the magnetic field created by the motion of the charge. We know that the magnetic field due to a moving charge is given by

$$\mathbf{B} = \frac{1}{c} \frac{q\mathbf{v} \times \mathbf{r}}{r^3}. \quad (10.27)$$

If we define the vector potential  $\mathbf{a}$  as

$$\mathbf{a} = \frac{q}{r} \frac{\mathbf{v}}{c}, \quad (10.28)$$

we can express  $\mathbf{B}$  in terms of  $\mathbf{a}$  as

$$\mathbf{B} = \nabla \times \mathbf{a}. \quad (10.29)$$

As we did for the scalar potential  $V$ , we argue that the correct formula for  $\mathbf{a}$  is

$$\mathbf{a}(\mathbf{R}, t) = q \frac{\mathbf{v}_{\text{ret}}/c}{r_{\text{ret}}(1 - \hat{\mathbf{r}}_{\text{ret}} \cdot \mathbf{v}_{\text{ret}}/c)}. \quad (10.30)$$

Equations (9.27) and (9.32) are known as the Liénard-Wiechert potentials.

The contribution to the electric field  $\mathbf{E}$  from  $V$  and  $\mathbf{a}$  is given by

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{a}}{\partial t}. \quad (10.31)$$

The derivatives in (9.33) are with respect to the observation coordinates. The difficulty associated with calculating these derivatives is that the potentials depend on  $t_{\text{ret}}$  which in turn depends on  $\mathbf{R}$ ,  $\mathbf{r}$ , and  $t$ . The result can be expressed as

$$\mathbf{E}(\mathbf{R}, t) = \frac{qr_{\text{ret}}}{(\mathbf{r}_{\text{ret}} \cdot \mathbf{u}_{\text{ret}})^3} [\mathbf{u}_{\text{ret}}(c^2 - v_{\text{ret}}^2) + \mathbf{r}_{\text{ret}} \times (\mathbf{u}_{\text{ret}} \times \mathbf{a}_{\text{ret}})], \quad (10.32)$$

where

$$\mathbf{u}_{\text{ret}} \equiv c\hat{\mathbf{r}}_{\text{ret}} - \mathbf{v}_{\text{ret}}. \quad (10.33)$$

The acceleration of the particle  $\mathbf{a}_{\text{ret}} = d\mathbf{v}(t)/dt|_{t=t_{\text{ret}}}$ . We also can show using (9.31) that the magnetic field  $\mathbf{B}$  is given by

$$\mathbf{B} = \hat{\mathbf{r}}_{\text{ret}} \times \mathbf{E}. \quad (10.34)$$

The above discussion is not rigorous, but we can accept (9.34) and (9.36) in the same spirit as we accept Coulomb's law and the Biot-Savart law. All of classical electrodynamics can be reduced

to (9.34) and (9.36), if we assume that the sources of all fields are charges, and all electric currents are due to the motion of charged particles. Note that (9.34) and (9.36) are consistent with the special theory of relativity and reduce to known results in the limit of stationary charges and steady currents.

Although (9.34) and (9.36) are deceptively simple (we do not even have to solve any differential equations), it is difficult to calculate the fields analytically even if the position of a charged particle is an analytic function of time. The difficulty is that we must find the retarded time  $t_{\text{ret}}$  from (9.26) for each observation position  $\mathbf{R}$  and time  $t$ . For example, consider a charged particle whose motion is sinusoidal, i.e.,  $x(t_{\text{ret}}) = A \cos \omega t_{\text{ret}}$ . To calculate the fields at the position  $\mathbf{R} = (X, Y, Z)$  at time  $t$ , we need to solve the following transcendental equation for  $t_{\text{ret}}$ :

$$t_{\text{ret}} = t - \frac{r_{\text{ret}}}{c} = t - \frac{1}{c} \sqrt{(X - A \cos^2 \omega t_{\text{ret}})^2 + Y^2 + Z^2}. \quad (10.35)$$

The solution of (9.37) can be expressed as a root finding problem for which we need to find the zero of the function  $f(t_{\text{ret}})$ :

$$f(t_{\text{ret}}) = t - t_{\text{ret}} - \frac{r_{\text{ret}}}{c}. \quad (10.36)$$

One way to find the root is to use a bracketing method. The idea is to first find a value  $t_a$  such that  $f(t_a) > 0$ , and another value  $t_b$  such that  $f(t_b) < 0$ . Because  $f(t_{\text{ret}})$  is continuous, there is a value of  $t_{\text{ret}}$  in the interval  $t_a < t_{\text{ret}} < t_b$  such that  $f(t_{\text{ret}}) = 0$ . There are various ways of guessing the solution and reducing the size of the interval. In the *bisection method* (see Section 6.6), we use the midpoint  $t_m = (t_a + t_b)/2$  as the guess, and compute  $f(t_m)$ . If  $f(t_m) > 0$ , find the next midpoint with  $t_a = t_m$ ; otherwise, find the next midpoint with  $t_b = t_m$ . Continue until  $f(t_m) = 0$  to the desired accuracy.

In the *method of false position*, we find the intersection of the line connecting the points  $(t_a, f(t_a))$  and  $(t_b, f(t_b))$  with the  $t_{\text{ret}}$  axis. The value of  $t_{\text{ret}}$  at the intersection,  $t_1$ , gives the first estimate for the zero of (9.37). If  $f(t_1) > 0$ , repeat the calculation with  $t_a = t_1$ , else repeat the calculation with  $t_b = t_1$ . This procedure is continued until the desired level of accuracy is achieved.

The bisection method and the method of false position are guaranteed to find a root. *Newton's method*, which uses the slope  $df(t_{\text{ret}})/dt_{\text{ret}}$  to estimate the solution, is much faster, but does not always give the correct answer and must be used with care. In this method the guess for  $t_{\text{ret}}$  at the  $n$ th iteration is given by

$$t_n = t_{n-1} - \frac{f(t_{n-1})}{df(t_{\text{ret}})/dt_{\text{ret}}|_{t_{\text{ret}}=t_{n-1}}}. \quad (10.37)$$

We use Newton's method unless the interval between two guesses is increasing. In that case we use the bisection method to decrease the interval. **Program radiation** uses this strategy to calculate the electric field due to an oscillating charge. We choose units such that the velocity is measured in terms of the speed of light  $c$ .

#### PROGRAM radiation

```
! compute electric fields due to accelerating charge and
! plot electric field lines
```

```

LIBRARY "csgraphics"
DIM snapshot$(100)
CALL initial(L,nsnap,radius,lpc,dt,dtheta)
FOR isnap = 1 to nsnap
    CALL draw_field(L,isnap,radius,dt,dtheta,snapshot$())
NEXT isnap
CALL animate(L,nsnap,snapshot$())
END

SUB initial(L,nsnap,a,lpc,dt,dtheta)
    LET L = 20
    CALL compute_aspect_ratio(2*L,xwin,ywin)
    SET WINDOW -xwin,xwin,-ywin,ywin
    LET a = 0.3                ! "radius" of visual image of charge
    LET lpc = 10              ! number of field lines
    LET dt = 0.4              ! time between plots
    LET dtheta = 2*pi/lpc
    LET nsnap = 15            ! number of snapshots
END SUB

SUB draw_field(L,isnap,a,dt,dtheta,snapshot$())
    ! adopted from Program fieldline
    DIM E(3),R(3),r_ret(3),v_ret(3),a_ret(3)
    DECLARE DEF dotproduct
    LET ds = 1
    LET t = isnap*dt          ! observation time
    LET R(1) = 0
    LET R(2) = 0
    CALL motion(R(),t,r_ret(),v_ret(),a_ret())
    ! find charge position at time t
    LET x = R(1) - r_ret(1)
    LET y = R(2) - r_ret(2)
    BOX CIRCLE x-a,x+a,y-a,y+a
    FOR theta = 0 to 2*pi step dtheta
        LET xline = x + a*cos(theta)
        LET yline = y + a*sin(theta)
        PLOT xline,yline;
    DO
        LET R(1) = xline
        LET R(2) = yline
        CALL field(R(),t,E()) ! compute fields
        LET Emag = sqr(dotproduct(E(),E()))
        ! new position on field line
        LET xline = xline + ds*E(1)/Emag
        LET yline = yline + ds*E(2)/Emag
        PLOT xline,yline;
    
```

```

        LOOP until abs(xline) > L or abs(yline) > L
        PLOT                               ! turn beam off
    NEXT theta
    BOX LINES -L,L,-L,L
    BOX KEEP -L,L,-L,L in snapshot$(isnap)
    CLEAR
END SUB

SUB motion(R(),t_ret,r_ret(),v_ret(),a_ret())
! compute motion of source
LET r_ret(1) = R(1) - 0.2*cos(t_ret)    ! harmonic motion
LET r_ret(2) = R(2)
LET r_ret(3) = R(3)
LET v_ret(1) = -0.2*sin(t_ret)        ! particle velocity
LET v_ret(2) = 0
LET v_ret(3) = 0
LET a_ret(1) = -0.2*cos(t_ret)        ! particle acceleration
LET a_ret(2) = 0
LET a_ret(3) = 0
END SUB

SUB field(R(),t,E())
! compute electric field vector
DECLARE DEF dotproduct
DIM r_ret(3),v_ret(3),a_ret(3),u_ret(3),uxa(3),w_ret(3)
CALL retarded_time(R(),t,t_ret)    ! find retarded time
! dynamical variables of moving charge
CALL motion(R(),t_ret,r_ret(),v_ret(),a_ret())
LET v2 = dotproduct(v_ret(),v_ret())
LET dist_ret = sqr(dotproduct(r_ret(),r_ret()))
FOR i = 1 to 3
    LET u_ret(i) = r_ret(i)/dist_ret - v_ret(i)
NEXT i
CALL crossproduct(u_ret(),a_ret(),uxa())
CALL crossproduct(r_ret(),uxa(),w_ret())
LET ru = dotproduct(r_ret(),u_ret())
LET E0 = dist_ret/ru^3
FOR i = 1 to 3
    LET E(i) = E0*(u_ret(i)*(1 - v2) + w_ret(i))
NEXT i
END SUB

SUB retarded_time(R(),t,t_ret)
LET tb = t                ! upper guess for retarded time
CALL fanddf(fb,dfdt,R(),t,tb)
LET ta = -1/1.6          ! lower guess for retarded time

```

```

! insure that f(ta) > 0 and f(tb) < 0
DO
  LET ta = ta*1.6
  CALL fanddf(fa,dfdt,R(),t,ta)
LOOP until fa > 0
LET t_ret = 0.5*(ta + tb)
CALL fanddf(f,dfdt,R(),t,t_ret)
CALL zero_of_f(f,dfdt,R(),t,t_ret,ta,tb)
END SUB

SUB fanddf(f,dfdt,R(),t,t_ret)
! calculate f and df/dt_r
DIM r_ret(3),v_ret(3),a_ret(3)
DECLARE DEF dotproduct
CALL motion(R(),t_ret,r_ret(),v_ret(),a_ret())
LET dist_ret = sqr(dotproduct(r_ret(),r_ret()))
LET f = t - t_ret - dist_ret
! derivative evaluated at retarded time
LET dfdt = -1 + dotproduct(r_ret(),v_ret())/dist_ret
END SUB

SUB zero_of_f(f,dfdt,R(),t,t_ret,ta,tb)
! do no more than 100 iterations to find the value of the reduced
! time t_ret such that f = t - t_ret - dist_ret = 0
LET eps = 1e-6
LET dt_r = tb - ta
FOR j = 1 to 100
  ! Newton difference between successive t_ret estimates
  LET dt = f/dfdt
  LET sign = ((t_ret - ta) - dt)*((t_ret - tb) - dt)
  LET dt_old = dt_r
  IF sign >= 0 or abs(2*dt) > abs(dt_old) then
    ! use bisection method if next Newton iteration is not
    ! between ta and tb, or if dt is not less than half
    ! of the old value of dt_r
    LET dt_r = 0.5*(ta - tb)
    LET t_ret = tb + dt_r
  ELSE
    ! use Newton's method
    LET dt_r = dt
    LET t_ret = t_ret - dt_r
  END IF
  IF abs(dt_r) < eps then EXIT SUB ! convergence test
  CALL fanddf(f,dfdt,R(),t,t_ret)
  IF f < 0 then
    LET tb = t_ret
  ELSE

```

```

        LET ta = t_ret
    END IF
NEXT j
PRINT "too many iterations"
STOP
END SUB

SUB animate(L,nsnap,snapshot$())
    FOR isnap = 1 to nsnap      ! show motion picture
        BOX SHOW snapshot$(isnap) at -L,-L
        PAUSE 1
    NEXT isnap
END SUB

SUB crossproduct(a(),b(),c())
    LET c(1) = a(2)*b(3) - a(3)*b(2)
    LET c(2) = a(3)*b(1) - a(1)*b(3)
    LET c(3) = a(1)*b(2) - a(2)*b(1)
END SUB

DEF dotproduct(a(),b())
    LET dotproduct = a(1)*b(1) + a(2)*b(2) + a(3)*b(3)
END DEF

```

*Problem 10.14.* Field lines from an accelerating charge

1. Read **Program radiation** carefully to understand the correspondence between the program and the calculation discussed in the text of the electric field lines due to an accelerating point charge. Describe qualitatively the nature of the electric field lines from an oscillating point charge.
2. Use **Program radiation** to calculate  $\mathbf{E}$  due to a positively charged particle oscillating about the origin according to  $x(t') = 0.2 \cos t'$ . The program draws field lines in the  $x$ - $y$  plane starting from a small circle surrounding the origin. Let the observation time be  $t = 1$  and stop drawing each field line when  $|x| > 20$  or  $|y| > 20$ . How do the field lines differ from those of a static charge at the origin?
3. What happens to the field lines as you increase the observation time  $t$ ?
4. One way of visualizing how the field lines change with time is by drawing the field lines at successive times, capturing the screen image each time. In True BASIC we can use the **BOX KEEP** statement and then show the images sequentially using the **BOX SHOW** statement. What new information does this mode of presentation convey?
5. Repeat the above observations for a charge moving with uniform circular motion about the origin.

*Problem 10.15.* Spatial dependence of radiating fields

1. As waves propagate from an accelerating point source, the total power that passes through a spherical surface of radius  $R$  remains constant. Because the surface area is proportional to  $R^2$ , the power per unit area or intensity is proportional to  $1/R^2$ . Also, because the intensity is proportional to  $E^2$ , we expect that  $E \propto 1/R$  far from the source. Modify *Program radiation* to verify this result for a charge that is oscillating along the  $x$ -axis according to  $x(t') = 0.2 \cos t'$ . Plot  $|E|$  as a function of the observation time  $t$  for a fixed position such as  $\mathbf{R} = (10, 10, 0)$ . The field should oscillate in time. Find the amplitude of this oscillation. Next double the distance of the observation point from the origin. How does the amplitude depend on  $R$ ?
2. Repeat part (a) for several directions and distances. Generate a polar diagram showing the amplitude as a function of angle in the  $x$ - $y$  plane. Is the radiation greatest along the line in which the charge oscillates?

*Problem 10.16.* Field lines from a charge moving at constant velocity

1. Use **Program radiation** to calculate  $\mathbf{E}$  due to a charged particle moving at constant velocity toward the origin, i.e.,  $x(t_{\text{ret}}) = 1 - 2t_{\text{ret}}$ . Take a snapshot at  $t = 0.5$  and compare the field lines with those you expect from a stationary charge.
2. Modify **SUB motion** so that  $x(t_{\text{ret}}) = 1 - 2t_{\text{ret}}$  for  $t_{\text{ret}} < 0.5$  and  $x(t_{\text{ret}}) = 0$  for  $t_{\text{ret}} > 0.5$ . Describe the field lines for  $t > 0.5$ . Does the particle accelerate at any time? Is there any radiation?

*Problem 10.17.* Frequency dependence of an oscillating charge

1. The radiated power at any point in space is proportional to  $E^2$ . Plot  $|E|$  versus time at a fixed observation point (e.g.,  $X = 10, Y = Z = 0$ ), and calculate the frequency dependence of the amplitude of  $|E|$  due to a charge oscillating at the frequency  $\omega$ . It is shown in standard textbooks that the power associated with radiation from an oscillating dipole is proportional to  $\omega^4$ . How does the  $\omega$ -dependence that you measured compare to that for dipole radiation? Repeat for a much bigger value of  $R$ , and explain any differences.
2. Repeat part (a) for a charge moving in a circle. Are there any qualitative differences?

## 10.5 \*Maxwell's Equations

In Section 9.4 we found that accelerating charges produce electric and magnetic fields which depend on position and time. We now investigate the direct relation between changes in  $\mathbf{E}$  and  $\mathbf{B}$  given by the differential form of Maxwell's equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c} \nabla \times \mathbf{E} \quad (10.38)$$

$$\frac{\partial \mathbf{E}}{\partial t} = c \nabla \times \mathbf{B} - 4\pi \mathbf{j}, \quad (10.39)$$

where  $\mathbf{j}$  is the electric current density. We can regard (9.40) and (9.41) as the basis of electrodynamics. In addition to (9.40) and (9.41), we need the relation between  $\mathbf{j}$  and the charge density  $\rho$  that expresses the conservation of charge:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}. \quad (10.40)$$

A complete description of electrodynamics requires (9.40), (9.41), and (??) and the initial values of all currents and fields.

For completeness, we obtain the Maxwell's equations that involve  $\nabla \cdot \mathbf{B}$  and  $\nabla \cdot \mathbf{E}$  by taking the divergence of (9.40) and (9.41), substituting (??) for  $\nabla \cdot \mathbf{j}$ , and then integrating over time. If the initial fields are zero, we obtain (using the relation  $\nabla \cdot (\nabla \times \mathbf{a}) = 0$  for any vector  $\mathbf{a}$ ):

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad (10.41)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (10.42)$$

If we introduce the electric and magnetic potentials, it is possible to convert the first-order equations (9.40) and (9.41) to second-order differential equations. However, the familiar first-order equations are better suited for numerical analysis. To solve (9.40) and (9.41) numerically, we need to interpret the curl and divergence of a vector. As its name implies, the curl of a vector measures how much the vector twists around a point. A coordinate free definition of the curl of an arbitrary vector  $\mathbf{W}$  is

$$(\nabla \times \mathbf{W}) \cdot \hat{\mathbf{S}} = \lim_{S \rightarrow 0} \frac{1}{S} \oint_C \mathbf{W} \cdot d\mathbf{l}, \quad (10.43)$$

where  $\mathbf{S}$  is the area of any surface bordered by the closed curve  $C$ , and  $\hat{\mathbf{S}}$  is a unit vector normal to the surface  $S$ .

Equation (??) gives the component of  $\nabla \times \mathbf{W}$  in the direction of  $\hat{\mathbf{S}}$  and suggests a way of computing the curl numerically. We divide space into cubes of linear dimension  $\Delta l$ . The rectangular components of  $\mathbf{W}$  can be defined either on the edges or on the faces of the cubes. We compute the curl using both definitions. We first consider a vector  $\mathbf{B}$  that is defined on the edges of the cubes so that the curl of  $\mathbf{B}$  is defined on the faces. (We use the notation  $\mathbf{B}$  because we will find that it is convenient to define the magnetic field in this way.) Associated with each cube is one edge vector and one face vector. We label the cube by the coordinates corresponding to its lower left front corner (see Figure ??a). The three components of  $\mathbf{B}$  associated with this cube are shown in Figure ??a. The other edges of the cube are associated with  $B$  vectors defined at neighboring cubes.

The discrete version of (??) for the component of  $\nabla \times \mathbf{B}$  defined on the front face of the cube  $(i, j, k)$  is

$$(\nabla \times \mathbf{B}) \cdot \hat{\mathbf{S}} = \frac{1}{(\Delta l)^2} \sum_{i=1}^4 B_i \Delta l_i, \quad (10.44)$$

where  $S = (\Delta l)^2$ , and  $B_i$  and  $l_i$  are shown in Figures ??b and ??c, respectively. Note that two of the components of  $\mathbf{B}$  are associated with neighboring cubes.

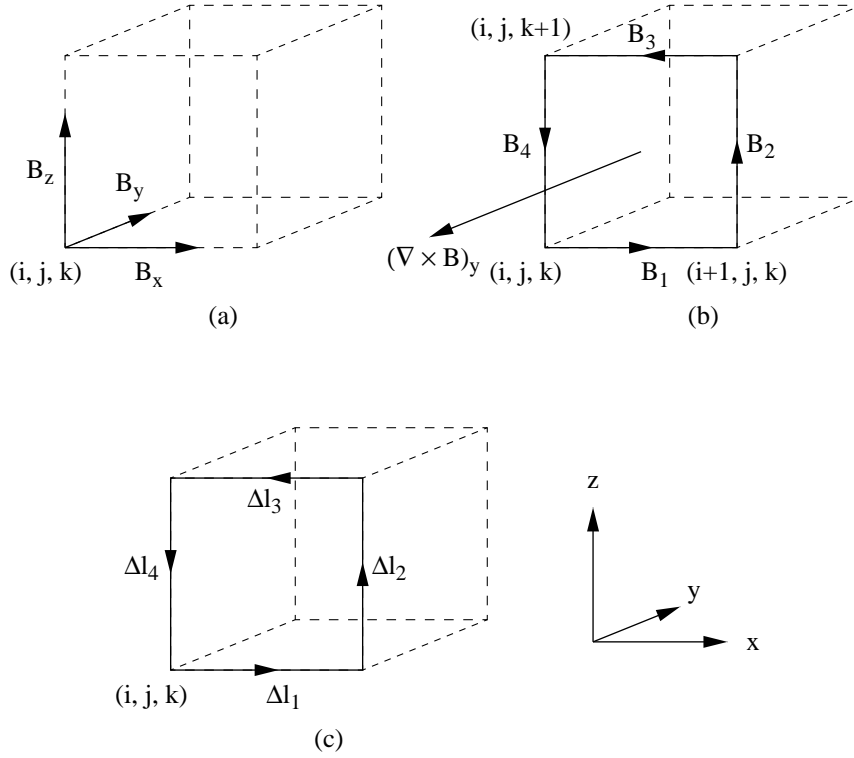


Figure 10.5: Calculation of the curl of  $\mathbf{B}$  defined on the edges of a cube. (a) The edge vector  $\mathbf{B}$  associated with cube  $(i, j, k)$ . (b) The components  $B_i$  along the edges of the front face of the cube.  $B_1 = B_x(i, j, k)$ ,  $B_2 = B_z(i + 1, j, k)$ ,  $B_3 = -B_x(i, j, k + 1)$ , and  $B_4 = -B_z(i, j, k)$ . (c) The vector components  $\Delta \mathbf{l}_i$  on the edges of the front face. (The  $y$ -component of  $\nabla \times \mathbf{B}$  defined on the face points in the negative  $y$  direction.)

The components of a vector also can be defined on the faces of the cubes. We call this vector  $\mathbf{E}$  because it will be convenient to define the electric field in this way. In Figure ??a we show the components of  $\mathbf{E}$  associated with the cube  $(i, j, k)$ . Because  $\mathbf{E}$  is normal to a cube face, the components of  $\nabla \times \mathbf{E}$  lie on the edges. The components  $E_i$  and  $l_i$  are shown in Figures ??b and ??c respectively. The form of the discrete version of  $\nabla \times \mathbf{E}$  is similar to (??) with  $B_i$  replaced by  $E_i$ , where  $E_i$  and  $l_i$  are shown in Figures ??b and ??c respectively. The  $z$ -component of  $\nabla \times \mathbf{E}$  is along the left edge of the front face.

A coordinate free definition of the divergence of the vector field  $\mathbf{W}$  is

$$\nabla \cdot \mathbf{W} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_{\mathbf{S}} \mathbf{W} \cdot d\mathbf{S}, \tag{10.45}$$

where  $V$  is the volume enclosed by the closed surface  $\mathbf{S}$ . The divergence measures the average flow of the vector through a closed surface. An example of the discrete version of (??) is given in (??).

We now discuss where to define the quantities  $\rho$ ,  $\mathbf{j}$ ,  $\mathbf{E}$ , and  $\mathbf{B}$  on the grid. It is natural to define the charge density  $\rho$  at the center of a cube. From the continuity equation (??), we see that this definition leads us to define  $\mathbf{j}$  at the faces of the cube. Hence, each face of a cube has a number associated with it corresponding to the current density flowing parallel to the outward normal to that face. Given the definition of  $\mathbf{j}$  on the grid, we see from (9.41) that the electric field  $\mathbf{E}$  and  $\mathbf{j}$  should be defined at the same places, and hence we define the electric field on the faces of the cubes. Because  $\mathbf{E}$  is defined on the faces, it is natural to define the magnetic field  $\mathbf{B}$  on the edges of the cubes. Our definitions of the vectors  $\mathbf{j}$ ,  $\mathbf{E}$ , and  $\mathbf{B}$  on the grid are now complete.

We label the faces of cube  $c$  by the symbol  $f_c$ . If we use the simplest finite difference method with a discrete time step  $\Delta t$  and discrete spatial interval  $\Delta x = \Delta y = \Delta z \equiv \Delta l$ , we can write the continuity equation as:

$$\left[\rho(c, t + \frac{1}{2}\Delta) - \rho(c, t - \frac{1}{2}\Delta t)\right] = -\frac{\Delta t}{\Delta l} \sum_{f_c=1}^6 j(f_c, t). \quad (10.46)$$

The factor of  $1/\Delta l$  comes from the area of a face  $(\Delta l)^2$  used in the surface integral in (??) divided by the volume  $(\Delta l)^3$  of a cube. In the same spirit, the discretization of (9.41) can be written as:

$$E(f, t + \frac{1}{2}\Delta t) - E(f, t - \frac{1}{2}\Delta t) = \Delta t [\nabla \times \mathbf{B} - 4\pi j(f, t)]. \quad (10.47)$$

Note that  $\mathbf{E}$  in (??) and  $\rho$  in (??) are defined at different times than  $\mathbf{j}$ . As usual, we choose units such that  $c = 1$ .

We next need to define a square around which we can discretize the curl. If  $\mathbf{E}$  is defined on the faces, it is natural to use the square that is the border of the faces. As we have discussed, this choice implies that we should define the magnetic field on the edges of the cubes. We write (??) as:

$$E(f, t + \frac{1}{2}\Delta t) - E(f, t - \frac{1}{2}\Delta t) = \Delta t \left[ \frac{1}{\Delta l} \sum_{e_f=1}^4 B(e_f, t) - 4\pi j(f, t) \right], \quad (10.48)$$

where the sum is over  $e_f$ , the four edges of the face  $f$  (see Figure ??b). Note that  $B$  is defined at the same time as  $j$ . In a similar way we can write the discrete form of (9.40) as:

$$B(e, t + \Delta t) - B(e, t) = -\frac{\Delta t}{\Delta l} \sum_{f_e=1}^4 E(f_e, t + \frac{1}{2}\Delta t), \quad (10.49)$$

where the sum is over  $f_e$ , the four faces that share the same edge  $e$  (see Figure ??b).

We now have a well defined algorithm for computing the spatial dependence of the electric and magnetic field, the charge density, and the current density as a function of time. This algorithm was developed by Yee, an electrical engineer, in 1966, and independently by Visscher, a physicist, in 1988 who also showed that all of the integral relations and other theorems that are satisfied by the continuum fields are also satisfied for the discrete fields.

Perhaps the most difficult part of the method is specifying the initial conditions since we cannot simply place a charge somewhere. The reason is that the initial fields appropriate for this

charge would not be present. Indeed, our rules for updating the fields and the charge densities reflect the fact that the electric and magnetic fields do not appear instantaneously at all positions in space when a charge appears, but instead evolve from the initial appearance of a charge. Of course, charges do not appear out of nowhere, but appear by disassociating the charges from neutral objects. Conceptually, the simplest initial condition corresponds to two charges of opposite sign moving oppositely to each other. This condition corresponds to an initial current on one face. From this current, a charge density and electric field appears using (??) and (??), respectively, and a magnetic field appears using (??).

Because we cannot compute the fields for an infinite lattice, we need to specify the boundary conditions. The easiest method is to use fixed boundary conditions such that the fields vanish at the edges of the lattice. If the lattice is sufficiently large, fixed boundary conditions are a reasonable approximation. However, fixed boundary conditions usually lead to nonphysical reflections off the edges, and a variety of approaches have been used including boundary conditions equivalent to a conducting medium that gradually absorbs the fields. In some cases physically motivated boundary conditions can be employed. For example, in simulations of microwave cavity resonators (see Problem ??), the appropriate boundary conditions are that the tangential component of  $\mathbf{E}$  and the normal component of  $\mathbf{B}$  vanish at the boundary.

As we have noted,  $\mathbf{E}$  and  $\rho$  are defined at different times than  $\mathbf{B}$  and  $\mathbf{j}$ . This “half-step” approach leads to well behaved equations that are stable over a range of parameters. An analysis of the stability requirement for the Yee-Visscher algorithm shows that the time step  $\Delta t$  must be smaller than the spatial grid  $\Delta l$  by:

$$c\Delta t \leq \frac{\Delta l}{\sqrt{3}}. \quad (\text{stability requirement}) \quad (10.50)$$

Your understanding of the Yee-Visscher algorithm for finding solutions to Maxwell’s equations will be enhanced by carefully reading the program listing for `Program maxwell` given in the following. The program uses a special True BASIC graphics subroutine `PICTURE arrow` that is called by a `DRAW` statement. Such a subroutine has arguments just like any other subroutine. Its utility is that it creates a graphical image that can then be rotated or shifted in space.

```
PROGRAM maxwell
! implementation of Yee-Visscher algorithm
LIBRARY "csgraphics"
PUBLIC E(0 to 21,0 to 21,0 to 21,3),B(0 to 21,0 to 21,0 to 21,3)
PUBLIC j(0 to 21,0 to 21,0 to 21,3)
PUBLIC n(3),dt,d1
PUBLIC mid,fpi,d1_1
CALL initial(yscale,byscale,jyscale)
CALL screen(#1,#2,#3)
DO
  CALL current(t)
  CALL newE
  CALL newB
  CALL plotfields(yscale,byscale,jyscale,#1,#2,#3)
DO
```

```

    LOOP until key input
    GET KEY k
LOOP until k = ord("s")
END

```

```

SUB initial(escale,bscale,jscale)
  DECLARE PUBLIC E(,,,),B(,,,),
  DECLARE PUBLIC n(),dt
  DECLARE PUBLIC mid,fpi,d1_1
  LET dt = 0.03
  LET n(1) = 8
  LET n(2) = 8
  LET n(3) = 8
  LET mid = n(1)/2
  LET d1 = 0.1
  LET d1_1 = 1/d1
  LET fpi = 4*pi
  LET escale = d1/(4*pi*dt)
  LET bscale = escale*d1/dt
  LET jscale = 1
  FOR x = 1 to n(1)
    FOR y = 1 to n(2)
      FOR z = 1 to n(3)
        FOR comp = 1 to 3
          LET E(x,y,z,comp) = 0
          LET B(x,y,z,comp) = 0
        NEXT comp
      NEXT z
    NEXT y
  NEXT x
END SUB

```

```

SUB current(t)
  DECLARE PUBLIC j(,,,),n(),mid
  ! steady current loop in x-y plane turned on at t = 0 and left on
  LET j(mid,mid,mid,2) = 1
  LET j(mid,mid,mid,1) = -1
  LET j(mid-1,mid,mid,2) = -1
  LET j(mid,mid-1,mid,1) = 1
END SUB

```

```

SUB newE
  ! E defined at the faces
  DECLARE PUBLIC E(,,,),B(,,,),j(,,,),
  DECLARE PUBLIC n(),dt,d1_1,fpi
  FOR x = 1 to n(1)

```

```

FOR y = 1 to n(2)
  FOR z = 1 to n(3)
    LET curlBx = B(x,y,z,2)+B(x,y+1,z,3)-B(x,y,z+1,2)-B(x,y,z,3)
    LET curlBx = curlBx*dl_1
    LET E(x,y,z,1) = E(x,y,z,1) + dt*(curlBx - fpi*j(x,y,z,1))
    LET curlBy = B(x,y,z,3)-B(x,y,z,1)+B(x,y,z+1,1)-B(x+1,y,z,3)
    LET curlBy = curlBy*dl_1
    LET E(x,y,z,2) = E(x,y,z,2) + dt*(curlBy - fpi*j(x,y,z,2))
    LET curlBz = B(x,y,z,1)+B(x+1,y,z,2)-B(x,y+1,z,1)-B(x,y,z,2)
    LET curlBz = curlBz*dl_1
    LET E(x,y,z,3) = E(x,y,z,3) + dt*(curlBz - fpi*j(x,y,z,3))
  NEXT z
NEXT y
NEXT x
END SUB

SUB newB
! B defined at the edges
DECLARE PUBLIC E(,,),B(,,)
DECLARE PUBLIC n(),dt,dl_1
FOR x = 1 to n(1)
  FOR y = 1 to n(2)
    FOR z = 1 to n(3)
      LET curlEx = E(x,y,z,3)-E(x,y,z,2)-E(x,y-1,z,3)+E(x,y,z-1,2)
      LET curlEx = curlEx*dl_1
      LET B(x,y,z,1) = B(x,y,z,1) - curlEx*dt
      LET curlEy = E(x,y,z,1)-E(x,y,z,3)-E(x,y,z-1,1)+E(x-1,y,z,3)
      LET curlEy = curlEy*dl_1
      LET B(x,y,z,2) = B(x,y,z,2) - curlEy*dt
      LET curlEz = E(x,y,z,2)-E(x,y,z,1)-E(x-1,y,z,2)+E(x,y-1,z,1)
      LET curlEz = curlEz*dl_1
      LET B(x,y,z,3) = B(x,y,z,3) - curlEz*dt
    NEXT z
  NEXT y
NEXT x
END SUB

SUB screen(#1,#2,#3)
DECLARE PUBLIC n()
LET L = n(1)
CALL compute_aspect_ratio(L+1,xwin,ywin)
SET BACKGROUND COLOR "black"
OPEN #1: screen 0,.5,0,.5
SET WINDOW 0,xwin,0,ywin
SET COLOR "white"
OPEN #2: screen 0.5,1,0.5,1

```

```

SET WINDOW 0,xwin,0,ywin
SET COLOR "white"
OPEN #3: screen 0.5,1,0,0.5
SET WINDOW 0,xwin,0,ywin
SET COLOR "white"
OPEN #4: screen 0,0.5,0.5,1
SET COLOR "white"
SET CURSOR 1,1
PRINT "Type s to stop"
PRINT
PRINT "Type any other key for next time step"
END SUB

```

```

SUB plotfields(escale,bscale,jscale,#1,#2,#3)
  DECLARE PUBLIC E(,,),B(,,),j(,,)
  DECLARE PUBLIC n(),dt,mid
  WINDOW #1
  CLEAR
  PRINT "E(x,y)"
  FOR x = 1 to n(1)
    FOR y = 1 to n(2)
      CALL plotarrow(E(x,y,mid,1),x,y,escale,0,0.5,0,pi)
      CALL plotarrow(E(x,y,mid,2),x,y,escale,0.5,0,pi/2,3*pi/2)
    NEXT y
  NEXT x
  WINDOW #2
  CLEAR
  PRINT "B(x,z)"
  FOR x = 1 to n(1)
    FOR z = 1 to n(3)
      CALL plotarrow(B(x,mid,z,1),x,z,bscale,0.5,0,0,pi)
      CALL plotarrow(B(x,mid,z,3),x,z,bscale,0,0.5,pi/2,3*pi/2)
    NEXT z
  NEXT x
  WINDOW #3
  CLEAR
  PRINT "j(x,y)"
  FOR x = 1 to n(1)
    FOR y = 1 to n(2)
      CALL plotarrow(j(x,y,mid,1),x,y,jscale,0,0.5,0,pi)
      CALL plotarrow(j(x,y,mid,2),x,y,jscale,0.5,0,pi/2,3*pi/2)
    NEXT y
  NEXT x
END SUB

```

```

SUB plotarrow(V,x,y,scale,shiftx,shifty,angle1,angle2)

```

```

IF V > 0 then
  DRAW arrow(V/scale) with rotate(angle1)*shift(x+shiftx,y+shifty)
ELSE IF V < 0 then
  DRAW arrow(-V/scale) with rotate(angle2)*shift(x+shiftx,y+shifty)
END IF
END SUB

PICTURE arrow(x)
  SET COLOR "yellow"
  PLOT LINES: -0.25*x,0;0.25*x,0;0.12*x,0.12*x
  PLOT LINES: 0.25*x,0;0.12*x,-0.12*x
  SET COLOR "white"
END PICTURE

```

*Problem 10.18.* Fields from a current loop

1. Program `maxwell` shows the electric field in the  $x$ - $y$  plane and the magnetic field in the  $x$ - $z$  plane in separate windows. The fields are represented by arrows, whose length is proportional to the field magnitude at each position where the field is defined. A steady current loop in the middle of the  $x$ - $y$  plane is turned on at  $t = 0$  and left on for all time (see SUB `current`). Before running the program, predict what you expect to see. Compare your expectations with the results of the simulation. Use  $\Delta t = 0.03$ ,  $\Delta l = 0.1$ , and take the number of cubes in each direction to be  $n(1) = n(2) = n(3) = 8$ .
2. Verify the stability requirement (??), by running your program with  $\Delta t = 0.1$  and  $\Delta l = 0.1$ . Then try  $\Delta t = 0.05$  and  $\Delta l = \Delta t\sqrt{3}$ . What happens to the results in part (a) if the stability requirement is not satisfied?
3. Modify the current density in part (a) so that  $\mathbf{j}$  is nonzero only for one time step. What happens to the electric and magnetic field vectors?
4. The amplitude of the fields far from the current loop should be characteristic of radiation fields for which the amplitude falls off as  $1/r$ , where  $r$  is the distance from the current loop to the observation point. Try to detect this dependence (if you have sufficient patience or computer resources).

*Problem 10.19.* Microwave cavity resonators

1. Cavity resonators are a practical way of storing energy in the form of oscillating electric and magnetic fields without losing as much energy as would be dissipated in a resonant LC circuit. Consider a cubical resonator of linear dimension  $L$  whose walls are made of a perfectly conducting material. The tangential components of  $\mathbf{E}$  and the normal component of  $\mathbf{B}$  vanish at the walls. Standing microwaves can be set up in the box of the form (cf. Reitz et al.)

$$E_x = E_{x0} \cos k_x x \sin k_y y \sin k_z z e^{i\omega t} \quad (10.51a)$$

$$E_y = E_{y0} \cos k_y y \sin k_x x \sin k_z z e^{i\omega t} \quad (10.51b)$$

$$E_z = E_{z0} \cos k_z z \sin k_x x \sin k_y y e^{i\omega t}. \quad (10.51c)$$

The wave vector  $\mathbf{k} = (k_x, k_y, k_z) = (m_x\pi/L, m_y\pi/L, m_z\pi/L)$ , where  $m_x$ ,  $m_y$ , and  $m_z$  are integers. A particular mode is labeled by the integers  $(m_x, m_y, m_z)$ . The initial electric field is perpendicular to  $\mathbf{k}$ , and  $\omega = ck$ . Implement the boundary conditions at  $(x = 0, y = 0, z = 0)$  and  $(x = L, y = L, z = L)$ . Set  $\Delta t = 0.05$ ,  $\Delta l = 0.1$ , and  $L = 1$ . At  $t = 0$ , set  $\mathbf{B} = 0$ ,  $\mathbf{j} = 0$  (there are no currents within the cavity), and use (??) with  $(m_x, m_y, m_z) = (0, 1, 1)$ , and  $E_{x0} = 1$ . Plot the field components at specific positions as a function of  $t$  and find the resonant frequency  $\omega$ . Compare your computed value of  $\omega$  with the analytical result. Do the magnetic fields change with time? Are they perpendicular to  $\mathbf{k}$  and  $\mathbf{E}$ ?

2. Repeat part (a) for two other modes.
3. Repeat part (a) with a uniform random noise added to the initial field at all positions. Assume the amplitude of the noise is  $\delta$  and describe the resulting fields for  $\delta = 0.1$ . Are they similar to those without noise? What happens for  $\delta = 0.5$ ? More quantitative results can be found by computing the power spectrum  $|E(\omega)|^2$  for the electric field at a few positions. What is the order of magnitude of  $\delta$  for which the maximum of  $|E(\omega)|^2$  at the standing wave frequency is swamped by the noise?
4. Change the shape of the container slightly by removing a  $0.1 \times 0.1$  cubical box from each of the corners of the original resonator. Do the standing wave frequencies change? Determine the standing wave frequency by adding noise to the initial fields and looking at the power spectrum. How do the standing wave patterns change?
5. Change the shape of the container slightly by adding a  $0.1 \times 0.1$  cubical box at the center of one of the faces of the original resonator. Do the standing wave frequencies change? How do the standing wave patterns change?
6. Cut a  $0.2 \times 0.2$  square hole in a face in the  $y$ - $z$  plane, and double the computational region in the  $x$  direction. Begin with a  $(0, 1, 1)$  standing wave, and observe how the fields “leak” out of the hole.

*Problem 10.20.* Billiard microwave cavity resonators

1. Repeat part (a) of Problem ?? for  $L_x = L_y = 2$ ,  $L_z = 0.2$ ,  $\Delta l = 0.1$ , and  $\Delta t = 0.05$ . Indicate the magnitude of the electric field in the  $L_z = 0.1$  plane by a color code. Choose an initial normal mode field distribution and describe the pattern that you obtain. Then repeat your calculation for a random initial field distribution.
2. Place an approximately circular conductor in the middle of the cavity of radius  $r = 0.4$ . Describe the patterns that you see. Such a geometry leads to chaotic trajectories for particles moving within such a cavity (see Project 6.24). Is there any evidence of chaotic behavior in the field pattern?
3. Repeat part (b) with the circular conductor placed off center.

## 10.6 Project

Much of the difficulty in understanding electromagnetic phenomena is visualizing its three-dimensional character. Although True BASIC has an excellent three-dimensional graphics toolkit, we have not

used it here because of the difficulty of translating its statements to other programming languages. Many interesting problems can be posed based on the simple, but nontrivial question of how the electromagnetic fields can best be represented visually in various contexts.

Many of the techniques used in this chapter, e.g., the random walk method and the relaxation method for solving Laplace's equation, have applications in other fields, especially problems in fluid flow and transport. Similarly, the multigrid method, discussed below, has far reaching applications.

*Project 10.21. Multigrid method*

In general, the relaxation method for solving Laplace's equation is very slow even using overrelaxation. The reason is that the local updates of the relaxation method cannot quickly take into account effects at very large length scales. The *multigrid method* greatly improves performance by using relaxation at many length scales. The important idea is to use a relaxation method to find the values of the potential on coarser and coarser grids, and then use the coarse grid values to determine the fine grid values. The fine grid relaxation updates take into account effects at short length scales. If we define the initial grid by a lattice spacing  $b = 1$ , then the coarser grids are characterized by  $b = 2^n$ , where  $n$  is the grid level. We need to decide how to use the fine grid values of the potential to assign values to a coarser grid, and then how to use a coarse grid to assign values to a finer grid. The first step is called prolongation and the second step is called restriction. There is some flexibility on how to do these two operations. We discuss one approach.

We define the points of the coarse grid as every other point of the fine grid. That is, if the set  $\{i, j\}$  represents the positions of the points of the fine grid, then  $\{2i, 2j\}$  represents the positions of the coarse grid points. The fine grid points that are at the same position as a coarse grid point are assigned the value of the potential of the corresponding coarse grid point. The fine grid points that have two coarse grid points as nearest neighbors are assigned the average value of these two coarse grid points. The other fine grid points have four coarse grid points as next nearest neighbors and are assigned the average value of these four coarse grid points. This prescription defines the restriction of the coarse grid to the fine grid.

In the full weighting prolongation method, each coarse grid point receives one fourth of the potential of the fine grid point at the same position, one eighth of the potential for the four nearest neighbor points of the fine grid, and one sixteenth of the potential for the four next nearest neighbor points of the fine grid. Note that the sum of these fractions,  $1/4 + 4(1/8) + 4(1/16)$ , adds up to unity. An alternative procedure, known as half weighting, ignores the next nearest neighbors and uses one half of the potential of the fine grid point at the same position as the coarse grid point.

1. Write a program that implements the multigrid method using Gauss-Seidel relaxation on a checkerboard lattice (see Problem 9.9b). In its simplest form the program should allow the user to intervene and decide whether to go to a finer or coarser grid, or to remain at the same level for the next relaxation step. Also have the program print the potential at each point of the current level after each relaxation step. Test your program on a  $4 \times 4$  grid whose boundary points are all equal to unity, and whose initial internal points are set to zero. Make sure that the boundary points of the coarse grids also are set to unity.
2. The exact solution for part (a) gives a potential of unity at each point. How many relaxation steps does it take to reach unity within 0.1% at every point by simply using the  $4 \times 4$  grid? How many steps does it take if you use one coarse grid and continue until the coarse grid values are within 0.1% of unity? Is it necessary to carry out any fine grid relaxation steps to

reach the desired accuracy on the fine grid? Next start with the coarsest scale, which is just one point. How many relaxation steps does it take now?

3. Repeat part (b), but change the boundary so that one side of the boundary is held at a potential of 0.5. Experiment with different sequences of prolongation, restriction, and relaxation.
4. Assume that the boundary points alternate between zero and unity, and repeat part (b). Does the multigrid method work? Should one go up and down in levels many times instead of staying at the coarsest level and then going down to the finest level?

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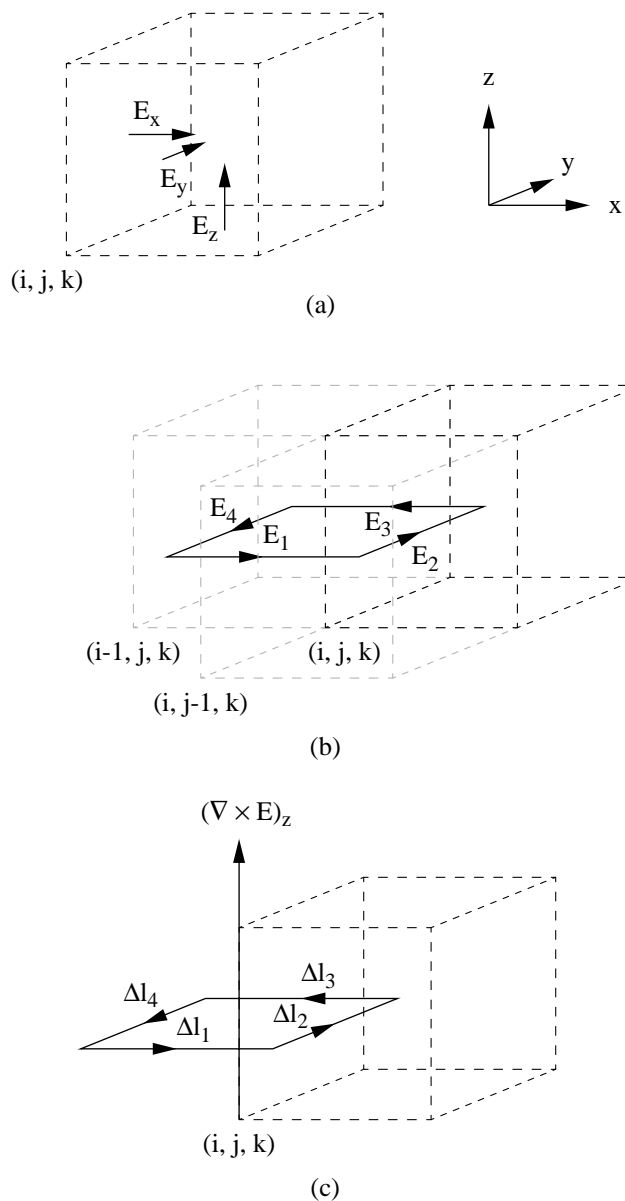


Figure 10.6: Calculation of the curl of the vector  $\mathbf{E}$  defined on the faces of a cube. (a) The face vector  $\mathbf{E}$  associated with the cube  $(i, j, k)$ . The components associated with the left, front, and bottom faces are  $E_x(i, j, k)$ ,  $E_y(i, j, k)$ ,  $E_z(i, j, k)$  respectively. (b) The components  $E_i$  on the faces that share the front left edge of the cube  $(i, j, k)$ .  $E_1 = E_x(i, j - 1, k)$ ,  $E_2 = E_y(i, j, k)$ ,  $E_3 = -E_x(i, j, k)$ , and  $E_4 = -E_y(i - 1, j, k)$ . The cubes associated with  $E_1$  and  $E_4$  also are shown. (c) The vector components  $\Delta l_i$  on the faces that share the left front edge of the cube. (The  $z$ -component of the curl of  $\mathbf{E}$  defined on the left edge points in the positive  $z$  direction.)