Problem 5.8. Consider an Ising chain of $N = 3$ spins and calculate $G(r)$ by exact enumeration of the 2^3 microstates. Choose free boundary conditions and calculate $G(r)$ using the microstates that you enumerated in Problem 5.5. Assume that the system is in equilibrium with a heat bath at temperature T and in zero magnetic field.

Solution. Because all spins are equivalent, we can calculate $G(r)$ with spin 1 as the origin.

microstate s	E_s	$s_1 s_2$	$s_1 s_3$
↑↑↑	-2	$+1$	$+1$
↑↑↓	0	0	-1
↑↓↑	2	-1	$+1$
↑↓↓	0	0	-1
↓↑↑	0	0	-1
↓↑↓	2	-1	$+1$
↓↓↑	0	0	-1
↓↓↓	-2	$+1$	$+1$

Table 1: Enumeration of states and their energies fpr the $N = 3$ Ising chain with free boundary conditions.

Because $\langle m \rangle = 0$ for the Ising chain we have $G(r) = \langle s_1 s_{r+1} \rangle$. From Table 1 we find that $Z = 2e^{2\beta} + 4e^0 + 2e^{-2\beta} = 2(e^{\beta} + e^{-\beta})^2$,

$$G(r=1) = \frac{1}{Z}\left[(1 \times e^{2\beta}) + 4 \times (0 \times e^0) + (-1 \times e^{-2\beta}) + (-1 \times e^{-2\beta}) + (1 \times e^{2\beta})\right] \tag{1a}$$

$$= \frac{2}{Z}\left[e^{2\beta} - e^{-2\beta}\right] = \frac{4}{Z}\sinh 2\beta = \operatorname{sech}\beta \tanh\beta \tag{1b}$$

and

$$G(r=2) = \frac{1}{Z}\left[2e^{2\beta} - 4e^0 + 2e^{-2\beta}\right] = \frac{2(e^{\beta} - e^{-\beta})^2}{2(e^{\beta} + e^{-\beta})^2} = 2\tanh^2\beta. \tag{1c}$$