12.2 Theorem *If $x \in H$ and $y \in H$, where H is an inner product space, then*

$$|(x, y)| \le \|x\| \|y\| \tag{1}$$

and

$$\|x + y\| \le \|x\| + \|y\|. \tag{2}$$

Moreover

$$\|y\| \le \|\lambda x + y\| \qquad \text{for every } \lambda \in \mathbb{C} \tag{3}$$

if and only if $x \perp y$.

PROOF. Put $\alpha = (x, y)$. A simple computation gives

$$0 \le \|\lambda x + y\|^2 = |\lambda|^2 \|x\|^2 + 2 \operatorname{Re}(\alpha\lambda) + \|y\|^2. \tag{4}$$

Hence (3) holds if $\alpha = 0$. If $x = 0$, (1) and (3) are obvious. If $x \neq 0$, take $\lambda = -\bar{\alpha}/\|x\|^2$. With this λ, (4) becomes

$$0 \le \|\lambda x + y\|^2 = \|y\|^2 - \frac{|\alpha|^2}{\|x\|^2}. \tag{5}$$

This proves (1) and shows that (3) is false when $\alpha \neq 0$. By squaring both sides of (2), one sees that (2) is a consequence of (1). ////

Note: Unless the contrary is explicitly stated, the letter H will from now on denote a Hilbert space.