Problem 1

Let $(\vec{a}, \vec{b}, \vec{c})$ be any three non-coplanar vectors. We define the "reciprocal" set $(\vec{A}, \vec{B}, \vec{C})$ as follows:

$$\vec{A} := \frac{1}{\omega}(\vec{b} \times \vec{c}), \quad (1)$$

$$\vec{B} := \frac{1}{\omega}(\vec{c} \times \vec{a}), \quad (2)$$

$$\vec{C} := \frac{1}{\omega}(\vec{a} \times \vec{b}), \quad (3)$$

where

$$\omega = \vec{a}.(\vec{b} \times \vec{c}) \quad (4)$$

is the volume of the parallelepiped formed from $(\vec{a}, \vec{b}, \vec{c})$.

(a) Calculate the dot product of each of $(\vec{a}, \vec{b}, \vec{c})$ with each of $(\vec{A}, \vec{B}, \vec{C})$.

(b) Consider the second rank tensor

$$\mathcal{T} = \vec{a}\vec{A} + \vec{b}\vec{B} + \vec{c}\vec{C}. \quad (5)$$

Calculate its scalar and its vector invariant.

(c) Show that $\mathcal{T}$ is idemfactor (unit tensor).

(d) Calculate the volume

$$\Omega = \vec{A}.(\vec{B} \times \vec{C}) \quad (6)$$

of the parallelepiped formed from $(\vec{A}, \vec{B}, \vec{C})$ in terms of the original set $(\vec{a}, \vec{b}, \vec{c})$. Can you express it purely in terms of ω?

(e) Calculate the reciprocal set $(\vec{\alpha}, \vec{\beta}, \vec{\gamma})$ of the reciprocal set $(\vec{A}, \vec{B}, \vec{C})$.

(f) Why was the original set $(\vec{a}, \vec{b}, \vec{c})$ restricted to be non-coplanar?

Problem 2

Without resorting to the use of components, show that the kinetic energy of a rotating rigid body

$$E_k = \frac{1}{2}\sum_i m_i \vec{v}_i.\vec{v}_i \quad (7)$$

can be written as

$$E_k = \frac{1}{2}\vec{\omega}.\mathcal{J}.\vec{\omega}, \quad (8)$$

where $\mathcal{J}$ is the moment of inertia tensor.

Solution of the problem 1:

(a) Using the definition of $\omega = \vec{a}.(\vec{b}\times\vec{c})$ we get

$$\vec{a}.\vec{A} = \vec{a}.\left[\frac{1}{\omega}(\vec{b}\times\vec{c})\right] = \boxed{1}.$$

Similarly goes the derivation of

$$\boxed{\vec{b}.\vec{B} = \vec{c}.\vec{C} = 1},$$

where we need cyclic permutation of triple scalar product. Finally the mixed terms

$$\boxed{\vec{a}.\vec{B} = \vec{a}.\vec{C} = \vec{b}.\vec{A} = \vec{b}.\vec{C} = \vec{c}.\vec{A} = \vec{c}.\vec{B} = 0},$$

where, in addition, we use the fact that the cross product is anti-commutative and thus for any vector $\vec{x}$ in 3D

$$\vec{x}\times\vec{x} = \vec{0}.$$

(b) Scalar identity or trace of our tensor of the rank 2 follows immediately using results from part (a)

$$SI(\mathcal{T}) := \vec{a}.\vec{A} + \vec{b}.\vec{B} + \vec{c}.\vec{C} = \boxed{3}.$$

In the case of vector identity we use definition of the reciprocal basis and observe cancelation of six terms (obtained from bac-cab rule)

$$\begin{aligned} VI(\mathcal{T}) &:= \vec{a}\times\vec{A} + \vec{b}\times\vec{B} + \vec{c}\times\vec{C} \\ &= \frac{1}{\omega}\left[\vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b}) + \vec{c}(\vec{b}.\vec{a}) - \vec{a}(\vec{b}.\vec{c}) + \vec{a}(\vec{c}.\vec{b}) - \vec{b}(\vec{c}.\vec{a})\right] \\ &= \boxed{\vec{0}}. \end{aligned}$$

(c) We need to show that for any vector $\vec{x}$ we have

$$\mathcal{T}.\vec{x} = \vec{x}.$$

Also, we should consider $\vec{x}.\mathcal{T}$. We can argue that our tensor is in some particular basis matrix 3×3 and such matrix is identity matrix if any vector is mapped to itself. More elegant way is to show that $\mathcal{T}.\mathcal{T} = \mathcal{T}$ so

$$\mathcal{T}.\mathcal{T} = (\vec{a}\vec{A} + \vec{b}\vec{B} + \vec{c}\vec{C}).(\vec{a}\vec{A} + \vec{b}\vec{B} + \vec{c}\vec{C})$$

and from part (a) immediately follows that only three terms do not vanish. Thus

$$\mathcal{T}.\mathcal{T} = \vec{a}(\vec{a}.\vec{A})\vec{A} + \vec{b}(\vec{b}.\vec{B})\vec{B} + \vec{c}(\vec{c}.\vec{C})\vec{C} = \mathcal{T}.$$

(d) The volume of the reciprocal basis relates to the volume of the original basis as follows

$$\Omega = \vec{A}.(\vec{B}\times\vec{C}) = \frac{1}{\omega^3}(\vec{b}\times\vec{c}).\left[(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\right].$$

Now we can use $bac - cab$ rule or

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}.(\vec{b}\times\vec{d})]\vec{c} - [\vec{a}.(\vec{b}\times\vec{c})]\vec{d},$$

which can be easily derived using Levi-Civita symbol ϵ_{ijk}. Note that $\vec{a}.(\vec{b}\times\vec{c}) \equiv \det(\vec{a}\ \vec{b}\ \vec{c}) \equiv [\vec{a},\vec{b},\vec{c}]$, where especially the last notation suggest invariance under the cyclic permutation. We than obtain

$$\Omega = \frac{1}{\omega^3}\overbrace{(\vec{b}\times\vec{c}).(\vec{a}}^{\omega}\overbrace{[\vec{a},\vec{b},\vec{c}]}^{\omega}) = \boxed{\frac{1}{\omega}}.$$

(e) The new reciprocal set is defined

$$\vec{\alpha} := \frac{1}{\Omega}(\vec{B} \times \vec{C}) = \omega \left[\frac{1}{\omega}(\vec{c} \times \vec{a}) \times \frac{1}{\omega}(\vec{a} \times \vec{b}) \right] = \boxed{\vec{a}},$$

where the last equality follows from part (d). Similarly, we find

$$\boxed{\vec{\beta} := \frac{1}{\Omega}(\vec{C} \times \vec{A}) = \vec{b}},$$

and

$$\boxed{\vec{\gamma} := \frac{1}{\Omega}(\vec{A} \times \vec{B}) = \vec{c}}.$$

(f) If the original set $(\vec{a}, \vec{b}, \vec{c})$ would be coplanar, i.e. linearly dependent set of vectors, they would span only two dimensional space (or one dimensional) and such space has three dimensional volume zero! It is easy to see this fact directly from the definition of the volume ω. We can immediately see that the definition of the reciprocal set would fail (division by zero).

Solution of the problem 2:

The velocity of the i^{th} rotating particle with angular velocity $\vec{\omega}$ is

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i.$$

Now we can use this expression and write

$$\begin{aligned} E_k &= \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i).(\vec{\omega} \times \vec{r}_i) \\ &= \frac{1}{2} \sum_i m_i \vec{\omega}.[\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)] \\ &= \frac{1}{2} \sum_i m_i \vec{\omega}.[\vec{\omega}(\vec{r}_i.\vec{r}_i) - \vec{r}_i(\vec{r}_i.\vec{\omega})] \\ &= \frac{1}{2} \sum_i m_i \vec{\omega}.[(\vec{r}_i.\vec{r}_i)\mathcal{I} - \vec{r}_i\vec{r}_i].\vec{\omega} \\ &= \frac{1}{2} \vec{\omega}. \underbrace{\sum_i m_i [(\vec{r}_i.\vec{r}_i)\mathcal{I} - \vec{r}_i\vec{r}_i]}_{\mathcal{J}} .\vec{\omega} \\ &= \boxed{\frac{1}{2} \vec{\omega}.\mathcal{J}.\vec{\omega}}, \end{aligned}$$

where $\mathcal{J}$ is the moment of inertia. It is symmetric tensor of 2^{nd} rank, i.e. it is represented by two indexes and in three dimensional space has nine components of which only six are independent. Note, that such matrix can be diagonalized by properly choosing the basis of the vector space.