Chapter 1: 13, 22, 23 : deadline Friday (09/15)

Problem 4

As we have seen in the class, the Lagrange's equations holds even for

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j}, \tag{1}$$

where $V = V(q_k, \dot{q}_k, t)$.

Show that under transformation $x_i = x_i(q_j, t)$ the functional form of our force is not changed, i.e.

$$F_i = Q_j \frac{\partial q_j}{\partial x_i} = -\frac{\partial V}{\partial x_i} + \frac{d}{dt}\frac{\partial V}{\partial \dot{x}_i}.$$

Also note that the equation (1) has the same functional form as Lagrange's equation of second kind! Just replace V with T or L. Note that you have just proved that Lagrange's equations are invariant under coordinate transformation.

In addition, observe what are the variables of these functions. How is your observation connected with the Newton's principle of determinacy?

Newton's principle of determinacy: The initial state of a mechanical system (the totality of position and velocities of its points at some moment of time) uniquely determines all of its motion, i.e. an applied forces are function only of position, velocity and time

$$m\ddot{\vec{x}} = \vec{F}(\vec{x}, \vec{v}, t).$$

Suggested problem

The Newton's equation of a charged particle in an electromagnetic field are (how many equations we are talking about?)

$$m\frac{d^2\vec{x}}{dt^2} = q(\vec{E} + \vec{v} \times \vec{B}).$$

Compare these equations (above) with the Lagrange's equations obtained from following Lagrangian

$$L = \frac{1}{2}m\vec{v}.\vec{v} - V(\vec{x}, \vec{v}, t).$$

Finaly, use gauge potentials $(\phi, \vec{A})$

$$\vec{B} = \nabla \times \vec{A}; \qquad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

to find

$$V(\vec{x}, \vec{v}, t) = q(\phi - \vec{v}.\vec{A}).$$

Note that

$$\vec{v} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{v}.\vec{A}) - (\vec{v}.\vec{\nabla})\vec{A},$$

and

$$\vec{A} = \frac{\partial}{\partial \vec{v}}(\phi - \vec{v}.\vec{A}).$$

Derive the following identities:

1. $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$
2. $\epsilon_{imk}\epsilon_{lmk} = 2\delta_{il}$
3. $\epsilon_{klm}\epsilon_{klm} = 6$
4. $\vec{v} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{v}.\vec{A}) - (\vec{v}.\vec{\nabla})\vec{A}$