Chapter 2: 13, 18, 25 : deadline Friday (09/29)

Problem 4

1. Show that d'Alembert principle can be written as

$$\delta T + \delta W = \frac{d}{dt}\left(\sum_i m_i \dot{\vec{r}}_i . \delta \vec{r}_i\right), \tag{1}$$

   where δT is a virtual kinetic energy and δW a virtual work.

2. The equation (1) is sometimes called the *central Lagrange's equation.* Express this equation in generalized coordinates q_j.

3. Explicitly transform virtual kinetic energy and virtual work. You should obtain

$$\sum_j \left[Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right]\delta q_j = 0. \tag{2}$$

   What are the arguments to obtain Lagrange's equations of second kind (with constraints and without constraints).

4. Go back to eq. (1) and integrate from t_1 to t_2. Deduce the Hamilton's principle.

Solution of the problem 4:

1. D'Alembert principle in the case of N particles can be written as

$$\sum_{i=1}^{N} (\vec{F}_i - m_i \ddot{\vec{r}}_i) . \delta \vec{r}_i = 0.$$

   The virtual work is

$$\delta W = \sum_{i=1}^{N} \vec{F}_i . \delta \vec{r}_i,$$

   thus we have to look closely at

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i . \delta \vec{r}_i .$$

In our case we know rhs of (1), which should be related to our term above. We can use product rule and $\frac{d}{dt}\delta\vec{r}_i = \delta\dot{\vec{r}}_i$ to get

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i\dot{\vec{r}}_i.\delta\vec{r}_i\right) = \sum_{i=1}^{N} m_i\ddot{\vec{r}}_i.\delta\vec{r}_i + \sum_{i=1}^{N} m_i\dot{\vec{r}}_i.\delta\dot{\vec{r}}_i = \sum_{i=1}^{N} m_i\ddot{\vec{r}}_i.\delta\vec{r}_i + \delta T,$$

where

$$\delta T := \delta\left[\frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{r}}_i.\dot{\vec{r}}_i\right] = \sum_{i=1}^{N} m_i\dot{\vec{r}}_i.\delta\dot{\vec{r}}_i.$$

2. In this problem we need to find the rhs of (1) in general coordinates q_j. We get

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i\dot{\vec{r}}_i.\delta\vec{r}_i\right) = \frac{d}{dt}\left(\sum_{j=1}^{3N}\frac{\partial T}{\partial\dot{x}_j}\delta x_j\right) = \frac{d}{dt}\left(\sum_{k=1}^{3N}\frac{\partial T}{\partial\dot{q}_k}\delta q_k\right).$$

Note that for the last equality we need to know

$$\frac{\partial T}{\partial\dot{x}_j} = \sum_{k=1}^{3N}\frac{\partial T}{\partial\dot{q}_k}\frac{\partial\dot{q}_k}{\partial\dot{x}_j},$$

(be sure you know why there are not included more partial derivatives - $T = T(q_k, \dot{q}_k, t)$ in general)

$$\delta x_j = \sum_{l=1}^{3N}\frac{\partial x_j}{\partial q_l}\delta q_l,$$

our well known relation between doted and not doted partials

$$\frac{\partial\dot{q}_j}{\partial\dot{x}_j} = \frac{\partial q_j}{\partial x_j},$$

and finally Jacobian of the coordinate transformation multiplied by its inverse producing identity matrix

$$\left(JJ^{-1}\right)_{kl} = \sum_{j=1}^{3N}\frac{\partial q_k}{\partial x_j}\frac{\partial x_j}{\partial q_l} = \delta_{kl}.$$

Thus we have made observation that the equation (1) is equivalent to

$$\delta T + \delta W = \frac{d}{dt}\left(\sum_{j=1}^{3N}\frac{\partial T}{\partial\dot{x}_j}\delta x_j\right),$$

and that can be written in the same form in any coordinate system

$$\delta T + \delta W = \frac{d}{dt}\left(\sum_{k=1}^{3N}\frac{\partial T}{\partial\dot{q}_k}\delta q_k\right). \tag{3}$$

3. Generalized kinetic energy depends on positions, velocities and time. The virtual change in the kinetic energy is not changing time, therefore

$$\delta T = \sum_{l=1}^{3N} \left(\frac{\partial T}{\partial q_l} \delta q_l + \frac{\partial T}{\partial \dot{q}_l} \delta \dot{q}_l \right) = \sum_{l=1}^{3N} \frac{\partial T}{\partial q_l} \delta q_l + \sum_{l=1}^{3N} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_l} \delta q_l \right) - \sum_{l=1}^{3N} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_l} \right) \delta q_l.$$

In the case of a virtual work we get

$$\delta W = \sum_{i=1}^{N} \vec{F}_i . \delta \vec{r}_i = \sum_{j,l=1}^{3N} F_j \frac{\partial x_j}{\partial q_l} \delta q_l = \sum_{l=1}^{3N} Q_l \delta q_l.$$

Substituting our results into the central Lagrange equation (3) we find the d'Alembert principle in generalized coordinates

$$\sum_{j=1}^{3N} \left[Q_j - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial T}{\partial q_j} \right] \delta q_j = 0.$$

Now, if the virtual displacements δq_j are linearly independent the Lagrange's equations of first kind follows

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j.$$

In the case of linearly dependent virtual displacements δq_j there are constraints. We know how to treat holonomic constraints $\phi_k(q_j, t) = 0$ and special case of nonholonomic constraints $\sum_{j=1}^{3N} a_{kj}(q_j, t) dq_j + a_{kt} dt = 0$. As in the lecture we drive

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \sum_{l=1}^{r} \lambda_l \frac{\partial \phi_l}{\partial q_j} + \sum_{k=1}^{s} \mu_k a_{kj} = Q_j,$$

where λ_l and μ_k are Lagrange's multipliers.

4. The integration of eq. (1) leads to

$$\int_{t_1}^{t_2} (\delta T + \delta W) \, dt = \left[\sum_i m_i \dot{\vec{r}}_i . \delta \vec{r}_i \right]_{t_1}^{t_2} .$$

If the configuration of the system is specified at times t_1 and t_2 such that the dynamical path and all imagined variations of this path coincide, then $\delta \vec{r}_i(t_1) = \delta \vec{r}_i(t_2) = 0$, and we get

$$\int_{t_1}^{t_2} (\delta T + \delta W) \, dt = 0,$$

or in the generalized coordinates, where $\delta q_i(t_1) = \delta q_i(t_2) = 0$,

$$\int_{t_1}^{t_2} (\delta T + \sum_{j=1}^{3N} Q_j \delta q_j) \, dt = 0.$$

These last two equations are often considered to be a generalized version of Hamilton's principle. It can also be considered an integral form of d'Alembert's principle.

If the external forces are conservative forces, $\vec{F}_i = -\nabla_i V$, then

$$\delta W = \sum_{j=1}^{3N} Q_j \delta q_j = \sum_{i,j} \vec{F}_i . \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = - \sum_{j=1}^{3N} \frac{\partial V}{\partial q_j} \delta q_j = -\delta V,$$

and we have

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0.$$

Finally, we assume that the integration and the virtual variation can be interchanged (can be done for holonomic system)

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = \delta \int_{t_1}^{t_2} (T - V)\, dt = \delta \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t)\, dt = 0.$$

This is Hamilton's principle for holonomic, conservative systems.