Chapter 3: 30 : deadline Friday (10/27)

## Problem 4

Use the Noether's theorem to explore:

1. Which condition must be satisfied by the Lagrange function $L(\vec{q}, \dot{\vec{q}}, t)$, and what conserved quatnities can be found, if the equations of motions are invariant under translations in time. Note that a translation in time is parametrized by the coordinate transformation $\delta q_i = \delta \dot{q}_i = 0$, $\delta t(t) = \delta\tau =$ constant.

2. Given the Lagrange function (in cartesian coordinates)

$$L = \frac{1}{2}m\dot{r}^2 - V(\vec{r})$$

   and the following transformations:

   (a) spatial translation

$$\delta x_1 = \delta x_2 = 0, \qquad \delta x_3 = const., \qquad \delta t = 0;$$

   (b) spatial rotation

$$\delta x_1 = -\delta\phi\, x_2, \qquad \delta x_2 = \delta\phi\, x_1, \qquad \delta x_3 = 0, \qquad \delta t = 0, \qquad (\delta\phi = const.);$$

   (c) Galilei transformations,

$$\delta x_1 = \delta x_2 = 0, \qquad \delta x_3 = \delta v_3\, t, \qquad \delta t = 0, \qquad (\delta v_3 = const.).$$

   Which conditions must hold, so that these are symmetry transformations? What are the conserved quantities?

## Solution of problem 1

In the class we have derived the condition that a mechanical system described by $L$ remains invariant under the infinitesimal symmetry transformation ($t' = t + \delta t(t)$ and $q_i' = q_i + \delta q_i(q_j, t)$)

$$\sum_i \left( \frac{\partial L}{\partial q_i} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \right) \delta q_i + \frac{\partial L}{\partial t} \delta t + \left( L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \frac{d}{dt} \delta t = -\frac{d}{dt} \delta f(q_j, t). \tag{1}$$

Note that (1) is the consequence of

$$-\delta L = L \frac{d}{dt} \delta t + \frac{d}{dt} \delta f(q_j, t). \tag{2}$$

In our case (1) reads

$$\frac{\partial L(q_j, \dot{q}_j, t)}{\partial t} \delta t = -\frac{d}{dt} \delta f(q_j, t),$$

where $\delta t = \delta \tau = const.$ Once we find $\delta f(q_j, t)$ we can use the result of Noether's theorem which give us the conserved quantity corresponding to the transformation and the Lagrange function $L$

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i + \left( L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \delta t + \delta f = \text{constant}.$$

In our case we formally (generally we do not know if the integral below exists) obtain

$$\boxed{\left( L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) - \int_{t_0}^{t} \frac{\partial L(q_j, \dot{q}_j, t)}{\partial t} \, dt = \text{constant}},$$

where the constant is determined from the initial conditions.

If $L$ is not explicitly dependent on time, we have very easy integral to deal with. Zero function has zero integral and we get our old result

$$E(q_j, \dot{q}_j) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = const.,$$

i.e. if the Lagrange function is not explicit function of time then the generalized energy $E$ is conserved (constant during the motion of the system). In addition, if the potential $V$ is not function of $\dot{q}_j$ and the kinetic energy $T(\dot{q}_j)$ is homogeneous function of second order, than the generalized energy becomes total energy of dynamical system

$$E(q_j, \dot{q}_j) = T(\dot{q}_j) + V(q_j) = const.$$

Solution of problem 2

For $\delta t = 0$ the conditional equation for a symmetry transformation (1) reads

$$\sum_i \left( \frac{\partial L}{\partial q_i} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \right) \delta q_i = -\frac{d}{dt} \delta f(q_j, t). \tag{3}$$

The left-hand side has to be a total derivative with respect to time, i.e. exists the following integral

$$\delta f(q_j, t) = -\int_{t_0}^{t} \sum_i \left( \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right) dt.$$

Than the quantity

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i - \int_{t_0}^{t} \sum_i \left( \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right) dt = \text{constant}$$

is conserved. In our case we have (note that generalized coordinates $q_i$ are cartesian coordinates in this case $x_i$)

$$\frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i}, \qquad \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i.$$

(a) For translational transformation the equation (3) reads

$$-\frac{\partial V(x_j)}{\partial x_3} \delta x_3 = -\frac{d}{dt} \delta f(x_j, t).$$

The left-hand side is not dependent on velocities $\dot{x}_i$, but right-hand side can be written as

$$-\frac{d}{dt} \delta f(x_j, t) = -\sum_i \frac{\partial}{\partial x_i} \delta f(x_j, t) \dot{x}_i - \frac{\partial}{\partial t} \delta f(x_j, t).$$

Thus we can conclude (note that the components of velocity are linearly independent)

$$\frac{\partial}{\partial x_i} \delta f(x_j, t) = 0 \quad \text{or} \quad \delta f(x_j, t) = \delta f(t).$$

Hence our symmetry condition is

$$\frac{\partial V(x_j)}{\partial x_3} \delta x_3 = \frac{d}{dt} \delta f(t).$$

It follows immediately (note the separation of variables - lhs is dependent only on position and rhs only on time) that $\frac{\partial V(x_j)}{\partial x_3} \delta x_3$ has to be constant (independent of position and time), and simple integration give us

$$\delta f(t) = \frac{\partial V(x_j)}{\partial x_3} \delta x_3 t + const.$$

If the above condition holds than the spatial translation is a symmetry transformation, and the conserved quantity is (after reduction by $\delta x_3$,which is arbitrary constant)

$$\boxed{m\dot{x}_3 + \frac{\partial V(x_j)}{\partial x_3} t = const.}$$

In particular, if the constant force in the $x_3$ direction vanishes, then

$$\frac{\partial V(x_j)}{\partial x_3} = 0,$$

and the conserved quantity is the generalized momentum

$$p_3 := \frac{\partial L}{\partial \dot{x}_3} = m\dot{x}_3.$$

Since now $\delta f = 0$, the Lagrange equation itself is invariant.

We thus have shown that the conservation of momentum follows from the spatial translational invariance of the Lagrange function, but not from the invariance of the equation of motion against spatial translations. In this general case the conserved quantity is

$$\tilde{\mathbf{P}} = \mathbf{p} - \mathbf{F}t,$$

implying that tme momentum is a linear function of time. Now $\mathbf{F}$ is a constant and homogeneous force field.

From a broader perspective, the constant force field illustrates the difference between local and global homogeneity of space. The field-filled space is locally homogeneous, because no point in space can be distinguished from any other by local measurements. However, the force field must be generated by some source, e.g. a distant mass for a gravity, or distant capacitor plates for a constant electric field. This source configuration destroys the global homogeneity of space.

(b) Here we have

$$\frac{\partial L}{\partial x_1}\delta x_1 + \frac{\partial L}{\partial x_2}\delta x_2 = -\frac{d}{dt}\delta f(x_j, t),$$

which for our Lagrange functions reads

$$\left(\frac{\partial V}{\partial x_1}x_2 - \frac{\partial V}{\partial x_2}x_1\right)\delta\phi = -\frac{d}{dt}\delta f(x_j, t).$$

The left-hand side is again (see part (a)) not dependent on velocities $\dot{x}_i$, but right-hand side can be written as

$$-\frac{d}{dt}\delta f(x_j,t) = -\sum_i \frac{\partial}{\partial x_i}\delta f(x_j,t)\dot{x}_i - \frac{\partial}{\partial t}\delta f(x_j,t).$$

Thus we can conclude (note that the components of velocity are linearly independent)

$$\frac{\partial}{\partial x_i}\delta f(x_j,t) = 0 \quad \text{or} \quad \delta f(x_j,t) = \delta f(t).$$

Hence our symmetry condition is

$$(\mathbf{r}\times\nabla V)_3\delta\phi = \frac{d}{dt}\delta f(t),$$

where $(\mathbf{r}\times\nabla V)_3$ is the third component of the torque. It follows immediately (note the separation of variables - lhs is dependent only on position and rhs only on time) that $(\mathbf{r}\times\nabla V)_3\delta\phi$ has to be constant (independent of position and time), and simple integration give us

$$\delta f(t) = (\mathbf{r}\times\nabla V)_3\, t\, \delta\phi + const.$$

If the above condition holds than the spatial translation is a symmetry transformation, and the conserved quantity is

$$(m\dot{x}_1\delta x_1 + m\dot{x}_2\delta x_2) + (\mathbf{r}\times\nabla V)_3\, t\, \delta\phi = const.$$

and after reduction by $\delta\phi$,which is an arbitrary constant, we get

$$\boxed{(\mathbf{r}\times\mathbf{p})_3 - (\mathbf{r}\times\nabla V)_3\, t = const.}$$

Now we can repeat the conclusion from part (a). If the torque is zero (i.e. we assume global isotropy of the space) then $\delta f(t) = 0$ and the Lagrange function is rotational invariant. The conserved quantity is the angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$.

(c) For Galileian transformation the invariance condition is

$$\left(\frac{\partial L}{\partial x_3} + \frac{\partial L}{\partial \dot{x}_3}\frac{d}{dt}\right)\delta v_3 t = -\frac{d}{dt}\delta f(x_j,t).$$

Using our Lagrange function we get

$$\left(-\frac{\partial V}{\partial x_3}t + m\dot{x}_3\right)\delta v_3 = -\frac{d}{dt}\delta f(x_j,t) = -\sum_i \frac{\partial}{\partial x_i}\delta f(x_j,t)\dot{x}_i - \frac{\partial}{\partial t}\delta f(x_j,t),$$

which leads to

$$\frac{\partial}{\partial x_1}\delta f(x_j,t) = \frac{\partial}{\partial x_2}\delta f(x_j,t) = 0, \qquad \frac{\partial}{\partial x_3}\delta f(x_j,t) = -m\delta v_3,$$

and

$$\frac{\partial}{\partial t}\delta f(x_j,t) = \frac{\partial V}{\partial x_3}\, t\, \delta v_3.$$

It follows from the first three equations that

$$\delta f(x_j,t) = -m x_3 \delta v_3 + g(t),$$

where $g(t)$ is an arbitrary function of time. Finally, we use the fourth equation

$$\frac{dg(t)}{dt} = \frac{\partial V}{\partial x_3}\, t\, \delta v_3$$

and find

$$\delta f(x_j,t) = -m x_3 \delta v_3 + \frac{1}{2}\frac{\partial V}{\partial x_3}\, t^2\, \delta v_3 + const.$$

The corespondent conserved quantity is

$$\boxed{m\dot{x}_3 t - m x_3 + \frac{1}{2}\frac{\partial V}{\partial x_3}\, t^2 = const.}$$

In the case of zero force $\frac{1}{2}\frac{\partial V}{\partial x_3} = 0$ we observe again, that the Lagrange function itself is invariant under the transformation. And the conserved quantity is

$$m\dot{x}_3 t - m x_3 = const.$$

That tells us that the free particle is moving in all coordinate systems connected by Galileian transformation with constant velocity (constant direction - straight line, and magnitude - speed). These systems, connected by Galilean transformation

$$x_i' = x_i + v_i t, \qquad t' = t,$$

are called inertial systems. We can not say in which inertial system we are based on any mechanics experiment, this is called the principle of relativity (non-relativistic). Note that the relativistic principle of relativity (Einstein's relativity) connects systems by Lorentz transformation

$$x_i' = x_i + (\frac{\gamma - 1}{v^2}\mathbf{v}.\mathbf{r} - \gamma t)v_i, \qquad t' = \gamma\left(t - \frac{\mathbf{v}.\mathbf{r}}{c^2}\right),$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and $c$ is the speed of the light.

We can see that in all cases (a)-(c) the local symmetry keeps the equation of motion invariant and the conserved quantity is directly related to the Newton's equations.

In the case of global symmetry we observe that the Lagrange function itself is invariant under the transformation. In that case we observe conservation of linear momentum, angular momentum and boost (tells us that the center of the mass is moving with a constant velocity). All these quantities are vector quantities, therefore we have conserved 9 quantities. In addition we have seen that translation in time leads to the conservation of energy, thus the total number of conserved quantities is 10!