1.1.2 POSITION, VELOCITY, AND ACCELERATION

The reason physicists are interested in making quantitative statements about the properties of a system is to compare theoretical predictions with experimental measurements. Among the most commonly measured quantities, particularly relevant to geometry and to the kinematic discussion of classical mechanics, is *distance*. The definition of distance is known as the *metric* of the space for which it is being defined. The Euclidean metric (i.e., in Euclidean space) defines the distance D between two points x and y, with coordinates x_i and y_i, as

$$D = \sqrt{\sum_i (x_i - y_i)^2}. \tag{1.5}$$

We will use this definition of distance also to discuss velocity.

The reason for bringing in the velocity at this point is that trajectories are usually found from other properties of the motion, of which velocity is an example. Another reason is that one is often interested not only in the trajectory itself, but also in other properties of the motion such as velocity.

If t is the time, the velocity $\mathbf{v}$ is defined as

$$\mathbf{v}(t) = \dot{\mathbf{x}}(t) \equiv \frac{d\mathbf{x}}{dt}. \tag{1.6}$$

(Here and in what follows, the dot ($\cdot$) over a symbol denotes differentiation with respect to t.) It is convenient to write $\mathbf{v}$ in terms of distance l along the trajectory. Let s be any parameter that increases smoothly and monotonically along the trajectory, and let $\mathbf{x}(s_0)$ and $\mathbf{x}(s_1)$ be any two points on the trajectory. Then the definition of Eq. (1.5) is used to define the distance along the trajectory between the two points as

$$l(s_0, s_1) = \int_{s_0}^{s_1} \left(\frac{dx_i}{ds}\frac{dx_i}{ds}\right)^{1/2} ds. \tag{1.7}$$

Note the use of the summation convention here: there is a sum over i. Although this definition of l seems to depend on the parameter s, it actually does not (see Problem 10). The trajectory can be parameterized by the time t or even by l itself, and the result would be the same.

If l is taken as the parameter, $\mathbf{v}$ can be written in terms of l:

$$\mathbf{v} = \frac{d\mathbf{x}}{dl}\frac{dl}{dt}. \tag{1.8}$$

But $d\mathbf{x}/dl$ is just the unit vector $\boldsymbol{\tau}$ tangent to the trajectory at time t. To see this, consider Fig. 1.1, which shows a section of a space curve. The tangent vector at the point $\mathbf{x}$ on the curve is in the direction of $\mathbf{T}$. The chord vector $\Delta\mathbf{x}$ between the points $\mathbf{x}$ and $\mathbf{x} + \Delta\mathbf{x}$ approaches parallelism to $\mathbf{T}$ in the limit as $\Delta\mathbf{x} \to 0$. In this limit the vector $\boldsymbol{\tau}$ can be expressed as

$$\boldsymbol{\tau} = \lim_{l \to 0} \frac{\Delta\mathbf{x}}{\Delta l} \equiv \frac{d\mathbf{x}}{dl}, \tag{1.9}$$

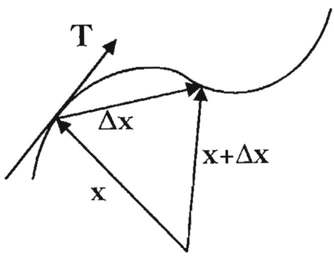

FIGURE 1.1
A vector **T** tangent to a space curve.

which is of unit length and parallel to **T**. Then (1.8) becomes

$$\mathbf{v} = \boldsymbol{\tau}\frac{dl}{dt} = \boldsymbol{\tau} v, \tag{1.10}$$

which says that **v** is everywhere tangent to the trajectory and equal in magnitude to the *speed* $v = \dot{l}$ along the trajectory.

Another important property of the motion is the acceleration, which is defined as the time derivative of the velocity, or

$$\mathbf{a} = \dot{\mathbf{v}} \equiv \ddot{\mathbf{x}} \equiv \frac{d\mathbf{v}}{dt}. \tag{1.11}$$

Then Eq. (1.10) implies that

$$\mathbf{a} = \frac{d\boldsymbol{\tau}}{dt}v + \boldsymbol{\tau}\frac{dv}{dt}. \tag{1.12}$$

The acceleration is related to the bending or curvature of the trajectory. To see this note first that $\dot{\boldsymbol{\tau}}$ is perpendicular to the trajectory. Indeed, $\boldsymbol{\tau}$ is a unit vector, so that $\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 1$, and therefore

$$\frac{d}{dt}(\boldsymbol{\tau} \cdot \boldsymbol{\tau}) = 0 = 2\boldsymbol{\tau} \cdot \frac{d\boldsymbol{\tau}}{dt}:$$

$d\boldsymbol{\tau}/dt$ is perpendicular to $\boldsymbol{\tau}$ and hence to the curve. Let **n** be the unit vector in this perpendicular direction, called the *principal normal vector*: $\mathbf{n} = \dot{\boldsymbol{\tau}}/|\dot{\boldsymbol{\tau}}|$. The *curvature* κ of the trajectory, the inverse of the *radius of curvature* ρ (see Problem 11) is defined by

$$\lim_{t_1 \to t_2} \frac{|\boldsymbol{\tau}(t_1) - \boldsymbol{\tau}(t_2)|}{|l(t_1) - l(t_2)|} \equiv \left|\frac{d\boldsymbol{\tau}}{dl}\right| = \kappa. \tag{1.13}$$

Thus $\kappa v = |\dot{\boldsymbol{\tau}}|$, or $\dot{\boldsymbol{\tau}} = \kappa v \mathbf{n}$. When these expressions for $\dot{l}$ and $\boldsymbol{\tau}$ are inserted into (1.12), one obtains

$$\mathbf{a} = \kappa v^2 \mathbf{n} + \ddot{l}\boldsymbol{\tau}. \tag{1.14}$$

In this expression the second term is the tangential acceleration and the first is the centrifugal acceleration (recall that $\kappa = 1/\rho$).

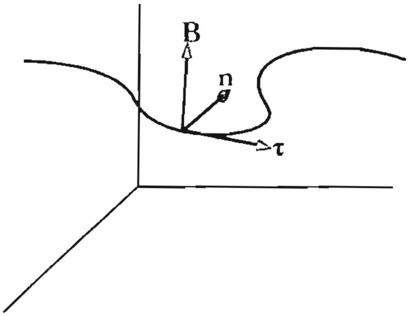

FIGURE 1.2
The tangent, normal, and binormal vectors to a space curve.

The acceleration lies in the plane formed by τ and $\mathbf{n}$, called the *osculating plane*. This can be thought of as the instantaneous plane of the curve; it is the plane that at each instant is being swept out by the tangent vector to the curve. The unit vector $\mathbf{B}$ normal to the osculating plane (Fig. 1.2), called the *binormal vector*, is given by (here the *wedge* $\wedge$ stands for the cross product, often denoted by $\times$)

$$\mathbf{B} = \tau \wedge \mathbf{n}.$$

Since $\dot{\tau}$ is parallel to $\mathbf{n}$ and $\mathbf{n}$ is a unit vector, the rate of change of $\mathbf{B}$ is parallel to $\mathbf{n}$. The torsion $\theta(t)$ of the curve is defined by writing $\dot{\mathbf{B}}$ in the form $\dot{\mathbf{B}} = -\theta \dot{l} \mathbf{n}$, or

$$\frac{d\mathbf{B}}{dl} = -\theta \mathbf{n}.$$

It can be shown (see Problem 11) that

$$\dot{\tau} = \kappa \dot{l} \mathbf{n},$$
$$\dot{\mathbf{n}} = -\kappa \dot{l} \tau + \theta \dot{l} \mathbf{B}.$$

These equations, known as the Frenet–Serret formulas, play an important role in the differential geometry of space curves and trajectories.

Of course, the first and the second time derivatives of $\mathbf{x}$ do not exhaust all possible properties of the motion; for instance one could ask about derivatives of higher order. From the purely mathematical point of view the study of these higher derivatives might be interesting, but physics essentially stops at the second derivative. This is because of Newton's second law $\mathbf{F} = m\mathbf{a}$, which connects the motion of a particle with the external forces acting on it in the real physical world. We therefore now turn from the mathematical treatment of the motion of point particles to a discussion of the physical principles that determine that motion. This section on kinematics has established the language in which we now state the axioms lying at the basis of classical mechanics, axioms that are justified ultimately by experiment.