

1.3 Noether's Theorem (for Improved Insight)

Noether's theorem, which we will now prove, says the following:

If the Euler-Lagrange equations of motion are invariant under a coordinate transformation $t, \mathbf{q} \rightarrow t'(t), \mathbf{q}'(\mathbf{q}, t)$, then there exists an integral of motion, i.e. a conserved quantity.

Given a Lagrange function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ of the coordinates q_i ($i = 1, \dots, l$) and time t , we introduce new coordinates t', \mathbf{q}' by defining

$$t' := t'(t) \quad , \quad q'_i := q'_i(\mathbf{q}, t) \quad . \quad (1)$$

This transformation shall be uniquely invertible. We can then write

$$t' := t + \delta t(t) \quad , \quad q'_i := q_i + \delta q_i(\mathbf{q}, t) \quad . \quad (2)$$

Initially the functions δt and δq_i are arbitrary. The velocities $\dot{\mathbf{q}}, \dot{\mathbf{q}}'$ are given by

$$\dot{q}_i := \frac{d}{dt} q_i \quad , \quad \dot{q}'_i := \frac{d}{dt'} q'_i \quad .$$

The connection between these two quantities reads as

$$\begin{aligned} \dot{q}'_i &= \frac{d}{dt'} q'_i = \frac{d}{dt} q'_i \frac{dt}{dt'} = \frac{d}{dt} (q_i + \delta q_i) \frac{dt}{dt'} \\ &= \left(\dot{q}_i + \frac{d}{dt} \delta q_i \right) \frac{1}{1 + (d/dt) \delta t} \quad , \end{aligned} \quad (3)$$

where we have used

$$\frac{dt}{dt'} = \frac{1}{dt'/dt} = \frac{1}{1 + (d/dt)\delta t} \quad (3')$$

For infinitesimal transformations this becomes

$$\delta \dot{q}_i := \dot{q}'_i - \dot{q}_i = \frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t \quad \checkmark \quad (4)$$

Since physics may not change under this coordinate transformation, the action has to remain invariant:

$$S(t_1, t_2) := \int_{t_1}^{t_2} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt = S'(t_1, t_2) := \int_{t'(t_1)}^{t'(t_2)} L'(\mathbf{q}'(t'), \dot{\mathbf{q}}'(t'), t') dt' \quad .$$

To achieve this aim the following must hold:

$$L'(\mathbf{q}', \dot{\mathbf{q}}', t) := L[\mathbf{q}(\mathbf{q}', t'), \dot{\mathbf{q}}(\mathbf{q}', \dot{\mathbf{q}}', t'), t(t')] \frac{dt}{dt'} \quad (5)$$

If the equations of motion are invariant in form under such coordinate transformation, we call this transformation a symmetry transformation. In the simplest case the Lagrange function itself remains invariant, i.e.

$$L'(\mathbf{q}', \dot{\mathbf{q}}', t') = L(\mathbf{q}', \dot{\mathbf{q}}', t') \quad ,$$

but this is not a necessity. It is sufficient that

$$L'(\mathbf{q}', \dot{\mathbf{q}}', t') = L(\mathbf{q}', \dot{\mathbf{q}}', t') + \frac{d}{dt'} \Omega(\mathbf{q}', t') \quad , \quad (6)$$

or in other words, that both Lagrange functions differ by a total derivative with respect to time. It is easily proved that for $\bar{L} = d[\Omega(\mathbf{q}, t)]/dt$, the equations of motion

$$\begin{aligned} \frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{q}_i} - \frac{\partial \bar{L}}{\partial q_i} &= \frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} \left(\sum_j \frac{\partial \Omega}{\partial q_j} \dot{q}_j + \frac{\partial \Omega}{\partial t} \right) - \frac{\partial}{\partial q_i} \left(\sum_j \frac{\partial \Omega}{\partial q_j} \dot{q}_j + \frac{\partial \Omega}{\partial t} \right) \\ &= \frac{d}{dt} \frac{\partial \Omega}{\partial q_i} - \left(\sum_j \frac{\partial^2 \Omega}{\partial q_i \partial q_j} \dot{q}_j + \frac{\partial^2 \Omega}{\partial t \partial q_i} \right) \\ &= \sum_j \frac{\partial^2 \Omega}{\partial q_i \partial q_j} \dot{q}_j + \frac{\partial^2 \Omega}{\partial t_i \partial q_i} - \left(\sum_j \frac{\partial^2 \Omega}{\partial q_i \partial q_j} \dot{q}_j + \frac{\partial^2 \Omega}{\partial t \partial q_i} \right) \\ &= 0 \quad , \end{aligned}$$

are fulfilled. Inserting (6) in (5), we get

$$L(\mathbf{q}(\mathbf{q}', t'), \dots, t(t')) \frac{dt}{dt'} = L(\mathbf{q}', \dot{\mathbf{q}}', t') + \frac{d}{dt'} \Omega(\mathbf{q}', t') \quad ;$$

and, reverting to the old coordinates,

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = L(\mathbf{q}'(\mathbf{q}, t), \dot{\mathbf{q}}'(\mathbf{q}, \dot{\mathbf{q}}, t), t'(t)) \frac{dt'}{dt} + \frac{d}{dt} \Omega(\mathbf{q}'(\mathbf{q}, t), t'(t)) \quad ,$$

$$1 + \frac{d}{dt} \delta t$$

which together with (2) or (3'), respectively, yields the equation

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) - L(\mathbf{q}'(\mathbf{q}, t), \dots, t'(t)) = L(\mathbf{q}'(\mathbf{q}, t), \dots) \frac{d}{dt} \delta t + \frac{d}{dt} \Omega(\mathbf{q}'(\mathbf{q}, t), t'(t)) \quad (7)$$

If the transformation is continuous, it is sufficient to consider infinitesimal transformations in (2). Then (7) may be written (to a first order approximation),

$$\begin{aligned} -\delta L &:= L(\mathbf{q}, \dot{\mathbf{q}}, t) - L(\mathbf{q} + \delta \mathbf{q}, \dot{\mathbf{q}} + \delta \dot{\mathbf{q}}, t + \delta t) \\ &= L(\mathbf{q}, \dot{\mathbf{q}}, t) \frac{d}{dt} \delta t + \frac{d}{dt} \Omega(\mathbf{q} + \delta \mathbf{q}, t + \delta t) \quad . \end{aligned}$$

In particular, if we choose $\delta \mathbf{q}, \delta t = 0$, then $\mathbf{q} = \mathbf{q}', t = t'$ and [from (6)] it follows that $d[\Omega(\mathbf{q}, t)]/dt = 0$. We may use this to rewrite $(-\delta L)$ as

$$\begin{aligned} -\delta L &= L \frac{d}{dt} \delta t + \frac{d}{dt} [\Omega(\mathbf{q} + \delta \mathbf{q}, t + \delta t) - \Omega(\mathbf{q}, t)] \\ &= L \frac{d}{dt} \delta t + \frac{d}{dt} \delta \Omega(\mathbf{q}, t) \quad . \end{aligned} \quad (8)$$

Now if

$$-\delta L = -\sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] - \frac{\partial L}{\partial t} \delta t \quad (9)$$

is inserted in (8) then, in view of (4),

$$\sum_i \left(\frac{\partial L}{\partial q_i} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \right) \delta q_i + \frac{\partial L}{\partial t} \delta t + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \frac{d}{dt} \delta t = -\frac{d}{dt} \delta \Omega(\mathbf{q}, t) \quad (10)$$

is, for arbitrary \mathbf{q}, t , the condition that a mechanical system described by L remains invariant under the infinitesimal symmetry transformation (2). In particular, if $\delta \Omega = 0, d(\delta t)/dt = 0$, then $\delta L = 0$ and the Lagrange function itself is invariant under the transformation. If (10) is fulfilled, then, using the equations of motion $\partial L/\partial q_i = d(\partial L/\partial \dot{q}_i)/dt$, it follows that

$$\begin{aligned} &\frac{d}{dt} \left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \delta t + \delta \Omega \right] \\ &= \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i + \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i - \frac{\partial L}{\partial q_i} \dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) \delta t \right] \\ &\quad + \frac{\partial L}{\partial t} \delta t + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \frac{d}{dt} \delta t + \frac{d}{dt} \delta \Omega = 0 \quad \checkmark \end{aligned}$$

i.e. the quantity

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \delta t + \delta \Omega = \text{const.} \quad (11)$$

is an integral of the motion (conserved quantity).