Motion of rigid body:

We start with the discrete rigid body which contains N masses. We will show how to go to the continuous limit later.

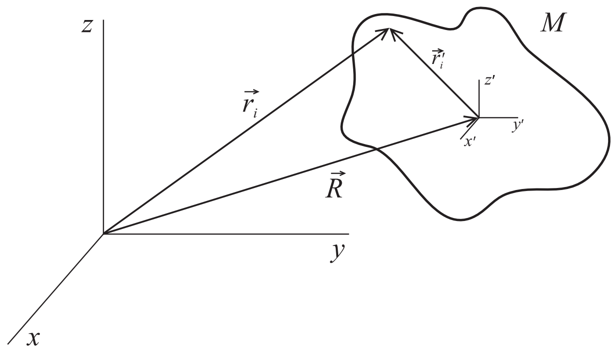

FIG. 1: Rigid body

Let $\vec{r}_i$ is the position of mass m_i in any given cartesian coordinate system. Than we can define the center of the mass

$$\vec{R} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i},$$

where $\sum_{i=1}^{N} m_i$ is the total mass M. Now we can write each vector $\vec{r}_i$ as (see Figure 1)

$$\vec{r}_i = \vec{R} + \vec{r}_i',$$

where $\vec{r}_i'$ is the position of the mass m_i in a coordinate system with it's origin at the center of mass, i.e. $\vec{R}' = \vec{0}$. Now we will use the Euler's theorem: *The general displacement of a rigid body with one point fixed is a rotation about some axis.* The fixed point is the center of mass in the primed coordinate system. The time derivative of vector $\vec{r}_i$ is the velocity needed for the total kinetic energy T

$$\dot{\vec{r}}_i = \dot{\vec{R}} + \dot{\vec{r}}_i' = \dot{\vec{R}} + \dot{\vec{\theta}} \times \vec{r}_i'$$

and

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i . \dot{\vec{r}}_i = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{R}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \, (\dot{\vec{\theta}} \times \vec{r}_i').(\dot{\vec{\theta}} \times \vec{r}_i') + \sum_{i=1}^{N} m_i \dot{\vec{R}}.(\dot{\vec{\theta}} \times \vec{r}_i').$$

The last term is zero as follows from the fact that the center of mass in the primed coordinate system is zero vector ($\vec{R}' = \vec{0}$)

$$\sum_{i=1}^{N} m_i \dot{\vec{R}}.(\dot{\vec{\theta}} \times \vec{r}_i') = (\dot{\vec{R}} \times \dot{\vec{\theta}}). \sum_{i=1}^{N} m_i \vec{r}_i' = (\dot{\vec{R}} \times \dot{\vec{\theta}}).M\vec{R}' = 0.$$

The next step is the second term in the total kinetic energy T. This one we have explored in the first homework! Here is the derivation: The velocity of the i^{th} rotating particle with angular velocity $\vec{\omega} \equiv \dot{\vec{\theta}}$ is

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i'.$$

Now we can use this expression and write

$$\begin{aligned}
E_k &= \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r}'_i).(\vec{\omega}\times\vec{r}'_i) \\
&= \frac{1}{2}\sum_i m_i\vec{\omega}.[\vec{r}'_i\times(\vec{\omega}\times\vec{r}'_i)] \\
&= \frac{1}{2}\sum_i m_i\vec{\omega}.[\vec{\omega}(\vec{r}'_i.\vec{r}'_i)-\vec{r}'_i(\vec{r}'_i.\vec{\omega})] \\
&= \frac{1}{2}\sum_i m_i\vec{\omega}.[(\vec{r}'_i.\vec{r}'_i)\mathcal{I}-\vec{r}'_i\vec{r}'_i].\vec{\omega} \\
&= \frac{1}{2}\vec{\omega}.\underbrace{\sum_i m_i[(\vec{r}'_i.\vec{r}'_i)\mathcal{I}-\vec{r}'_i\vec{r}'_i]}_{\mathcal{J}}.\vec{\omega} \\
&= \boxed{\frac{1}{2}\vec{\omega}.\mathcal{J}.\vec{\omega}},
\end{aligned}$$

where $\mathcal{J}$ is the moment of inertia. It is symmetric tensor of 2^{nd} rank, i.e. it is represented by two indexes and in three dimensional space has nine components of which only six are independent.

Now we can write the total kinetic energy

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\vec{\omega}.\mathcal{J}.\vec{\omega},$$

where the first term is the kinetic energy of the center of mass and the second term is the rotational motion around axis going through the center of mass. Only in the case that the center of mass is moving along straight line and the axis of the rotation is the principal one and not time dependent (i.e. does not change the orientation during the motion) we can state that the we have separated the translational motion and the rotational motion and write

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mathcal{J}\omega^2.$$

Finally note that the continuous case has the following modifications: The center of mass

$$\vec{R} = \frac{\int_M \vec{r}(m)\,dm}{M} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{M}$$

and the components of the moment of inertia tensor

$$\mathcal{J}_{ij} = \int_V \rho(\vec{r}\,')(\delta_{ij}r'^2 - r'_i r'_j)\,dV.$$