3.6 TIME-VARYING MASS SYSTEM (ROCKET MOTION)

We conclude this chapter by considering the motion of a time-varying mass system, such as a rocket. The time variation of the mass in rocket motion is due to the expulsion of the exhaust. This is totally different from the case in which the mass of an object varies due to its high speed (the relativistic effect).

Because of its special interest, we shall study the motion of a rocket in the short time interval during which its fuel is being consumed. At time t the rocket of mass m is moving with a velocity v relative to a fixed coordinate system (an inertial system), as illustrated in Figure 3.15. The exhaust is ejected with a velocity $-u$ relative to the rocket, assuming that both the force f_g of the expelled gas and the jet velocity $-u$ are constant. The velocity of the exhaust relative to the fixed coordinate system is $(v - u)$. An infinitesimal amount of time dt later the rocket's mass has decreased to $m + dm$, where $-dm$ is the mass of the exhaust ejected between t and $t + dt$. At time $t + dt$, the velocity of the rocket is $v + dv$, and the velocity of the exhaust relative to the fixed coordinate system is $(v + dv - u)$. Applying the impulse-momentum equation to the exhaust

$$-f_g dt = -dm(v + dv - u) \tag{3.62}$$

where $-f_g dt$ is the impulse due to gas force f_g acting on exhaust mass ($f_g >>$ gravitational force F_g on gas).

The impulse-momentum equation for the rocket is

$$F^{(e)}dt + f_g dt = (m + dm)(v + dv) - mv$$

or

$$F^{(e)}dt + f_g dt = mdv + dmdv + vdm \tag{3.63}$$

where the external forces $F^{(e)}$ are primarily due to gravity and air resistance (the aerodynamic drag).

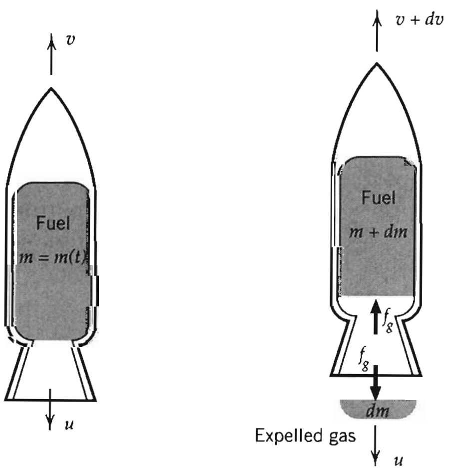

Figure 3.15 Motion of a rocket.

Adding Eqs. 3.62 and 3.63, we obtain, to first order in the differentials,

$$F^{(e)}dt = mdv + udm$$

from which we obtain the fundamental equation of rocket motion

$$m\frac{dv}{dt} = F^{(e)} - u\frac{dm}{dt} \qquad \left(\frac{dm}{dt} < 0\right) \tag{3.64}$$

where udm/dt is often referred to as the rocket thrust. The exhaust velocity u depends on the type of rocket fuel that is burned. Fuels with low molecular weights generally have higher exhaust velocities, and thus yield high rocket velocities. Present rocket technology gives exhaust velocities that are close to the thermodynamic limit for chemical fuels.

If $F^{(e)} = 0$, as would be the case in free space, Eq. 3.64 can be readily solved. Multiplying through by dt, we obtain

$$dv = -u\frac{dm}{m} \tag{3.65}$$

The result of integration is

$$v_f - v_i = u \ln\frac{m_i}{m_f} \tag{3.66}$$

where subscripts i and f represent initial and final values, respectively.

For rockets fired vertically upward from the surface of the earth, the rocket Eq. 3.65 becomes

$$m\frac{dv}{dt} = -mg - u\frac{dm}{dt} \tag{3.67}$$

where air resistance is neglected and the earth is regarded as an inertial frame. An analytic solution to this equation is possible for a constant rate of burn,

$$\frac{dm}{dt} = -\alpha m_i$$

from which

$$m = m_i(1 - \alpha t) \tag{3.68}$$

where α is a constant factor. Substituting Eq. 3.68 into Eq. 3.67 and integrating from ignition at $t = 0$ to burnout at $t = t_f$

$$\int_0^{v_f} dv = \int_0^{t_f} \frac{-g + \alpha u}{1 - \alpha t}\, dt$$

we obtain the velocity at the instant the fuel becomes exhaust

$$v_f = -gt_f - u\ln(1 - \alpha t_f) \tag{3.69}$$

If the initial velocity is zero at $t = 0$, the height reached at time t_f is then

$$h_f = ut_f - \frac{gt_f^2}{2} + \frac{u}{\alpha}(1 - \alpha t_f)\ln(1 - \alpha t_f) \qquad (3.70)$$

Burnout time t_f can be expressed in terms of m_i and m_f, from Eq. 3.68, as

$$t_f = \frac{1}{\alpha}(-m_f/m_i + 1)$$

Thus the final velocity v_f can be written

$$v_f = -\frac{g}{\alpha}\left(1 - \frac{m_f}{m_i}\right) + u \ln \frac{m_i}{m_f} \qquad (3.71)$$

and the height h_f is

$$h_f = \frac{1}{\alpha}\left(1 - \frac{m_f}{m_i}\right)\left[u - \frac{g}{2\alpha}\left(1 - \frac{m_f}{m_i}\right)\right] + \frac{um_f}{\alpha m_i} \ln \frac{m_f}{m_i} \qquad (3.72)$$

The actual trajectory of a rocket has considerable curvature. For the Apollo moon rocket (a three-stage Saturn V rocket), the space vehicle is 93 km down range at an altitude of 67 km when the first stage burns out. The velocity v_f attained would be higher than that calculated from Eq. 3.71 for a vertical trajectory, as we should expect.

Problems

3.1 A particle of mass m, which at time $t = 0$ has position x_0 and velocity v_0, is being acted upon by a sinusoidal force

$$F = F_0 \sin(\omega t - \phi)$$

where F_0, ω, and ϕ are constants. Find the equations for position and velocity for all positive time t.

3.2 A particle of mass m is whirled on the end of a string of length R. The motion is in a vertical plane in the earth's gravitational field. The instantaneous speed is v when the string makes angle θ with the horizontal. Find the tension T in the string and the tangential acceleration at this instant.

3.3 A uniform rope of length L is pulled by gravity from a smooth table. Assuming that a length L_0 initially hangs over the table and that the rope starts from rest, find x as a function of t for $L_0 < x < L$, where x represents the distance from the lower end of the rope to the table top. The height of the table is H.

3.4 Mass M hangs from a string of length L that is attached to a rod rotating at constant angular frequency ω, as shown in Figure 3.16. The mass moves with