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chapter 9:

problem 4: (solution):

The Jacobian of the transformation is $M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$

which is, $M = \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix}$

Hence

$$\tilde{M} J M = \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{q} & \cot p \\ \cot p & -q\csc^2 p \end{pmatrix} \begin{pmatrix} \cot p & -q\csc^2 p \\ \frac{1}{q} & -\cot p \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \csc^2 p - \cot^2 p \\ \cot^2 p - \csc^2 p & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

So the symplectic condition is satisfied $\Rightarrow$ The transformation is canonical.

problem 5 : (Solution)

The Jacobian of the transformation is $M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$

which is, $M = \begin{pmatrix} \left(\frac{\alpha}{p}\right) \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} & -\left(\frac{\alpha q}{p^2}\right) \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} \\ \alpha q & \frac{p}{\alpha} \end{pmatrix}$

So, $$\tilde{M} J M = \begin{pmatrix} \left(\frac{\alpha}{p}\right) \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} & \alpha q \\ -\left(\frac{\alpha q}{p^2}\right) \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} & \frac{p}{\alpha} \end{pmatrix} \begin{pmatrix} \alpha q & \frac{p}{\alpha} \\ -\left(\frac{\alpha}{p}\right) \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} & \left(\frac{\alpha q}{p^2}\right) \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

So the symplectic condition is satisfied $\Rightarrow$ the transformation is canonical.

problem 6: (Solution)

(a) The Jacobian of the transformation is

$$M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{2}\right) \frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \\ q^{-1/2}\sin p + 2\cos p \sin p & 2q^{1/2}\cos p + 2q\cos^2 p - 2q\sin^2 p \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1}{2}\right) \frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \\ q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix}$$

thus, $$\tilde{M} J M = \begin{pmatrix} \left(\frac{1}{2}\right) \frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & q^{-1/2}\sin p + \sin 2p \\ -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix}$$

$$\times \begin{pmatrix} q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \\ -\left(\frac{1}{2}\right) \frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \end{pmatrix}$$

which is,

$$\tilde{M} J M = \begin{pmatrix} 0 & \dfrac{\cos^2 p + \sin^2 p + q^{1/2}\cos p \cos 2p + q^{1/2}\sin p \sin 2p}{1 + q^{1/2}\cos p} \\ -\left(\dfrac{\cos^2 p + \sin^2 p + q^{1/2}\cos p \cos 2p + q^{1/2}\sin p \sin 2p}{1 + q^{1/2}\cos p}\right) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

So the symplectic condition is satisfied $\Rightarrow$ the transformation is canonical

(b) For an F_3 function the relevant relations are $q = -\dfrac{\partial F}{\partial p}$, $P = -\dfrac{\partial F}{\partial Q}$.

We have $F_3(p, Q) = -(e^{Q} - 1)^2 \tan p$

so,
$$P = -\frac{\partial F_3}{\partial Q} = 2e^{Q}(e^{Q} - 1)\tan p$$
$$q = -\frac{\partial F_3}{\partial p} = (e^{Q} - 1)^2 \sec^2 p.$$

The second of these may be solved to yield Q in terms of q and p:

$$Q = \log\left(1 + q^{1/2}\cos p\right)$$

and then we may plug this back into the equation for P to obtain

$$P = 2q^{1/2}\sin p + q\sin 2p$$

Problem 24; (solution):

ⓐ The Jacobian of the transformation is $M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$, which is

$$M = \begin{pmatrix} ia & 1 \\ -\frac{1}{2} & \frac{1}{2ia} \end{pmatrix} \Rightarrow \tilde{M} J M = \begin{pmatrix} ia & -\frac{1}{2} \\ 1 & \frac{1}{2ia} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} ia & 1 \\ -\frac{1}{2} & \frac{1}{2ia} \end{pmatrix}$$

which is, $$\tilde{M} J M = \begin{pmatrix} ia & -\frac{1}{2} \\ 1 & \frac{1}{2ia} \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2ia} \\ -ia & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J$$

chapter 10

problem 6 : (solution)

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{q}{c}(\vec{v}\cdot\vec{A}) - \frac{k}{2}(x^2+y^2)$$

Inserting the vector potential $\vec{A} = \frac{1}{2}B(-y\hat{i} + x\hat{j})$, then

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{qB}{2c}(x\dot{y} - y\dot{x}) - \frac{k}{2}(x^2+y^2)$$

Now, we go over to polar coordinates according to

$$x = r\cos\theta \rightarrow \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$y = r\sin\theta \rightarrow \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

and obtain, $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{qB}{2c}r^2\dot{\theta} - \frac{k}{2}r^2$

To go over to Hamiltonian we introduce the canonical momenta:

$$P_r = \frac{\partial L}{\partial \dot r} = m\dot r$$

$$P_\theta = \frac{\partial L}{\partial \dot \theta} = mr^2\dot\theta + \frac{qB}{2c} r^2$$

Then the Hamiltonian is, $H = P_r \dot r + P_\theta \dot\theta - L$

$$= \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) + \frac{1}{2}kr^2$$

which is, $H = \frac{1}{2m} P_r^2 + \frac{1}{2mr^2}\left(P_\theta - \frac{qB}{2c} r^2\right)^2 + \frac{1}{2}kr^2$

The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta} - \frac{qB}{2c} r^2\right)^2 + \frac{1}{2}kr^2 \pm \frac{\partial S}{\partial t} = 0 \qquad (*)$$

since θ is cyclic its corresponding conjugate momentum must be constant, and we look for a solution of the form

$$S(r,\theta,E,\alpha,t) = f(r,E,\alpha) + \alpha\theta - Et$$

Equation (*) becomes ✓

$$\frac{1}{2m}\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\alpha - \frac{qB}{2c}r^2\right)^2 + \frac{1}{2}kr^2 = E$$

with formal solution

$$f(r) = \int dr \sqrt{2mE - mkr^2 - \frac{1}{r^2}\left(\alpha - \frac{qB}{2c}r^2\right)^2}$$

If $\alpha = 0$ this simplifies to

$$f(r) = \int dr \sqrt{2mE - m^2(\omega_0^2 + \omega_c^2)r^2}$$

and the problem becomes just that of the normal harmonic oscillator, with frequency

$$\omega = \sqrt{\omega_0^2 + \omega_c^2} \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad \omega_c = \frac{qB}{2mc}$$