Laboratory 8: The L-R-C Circuit and Resonance

Sect. 1: Review of the Forced, Damped Oscillator. It has some time since we looked at the forced, damped oscillator and, perhaps, a review is appropriate. Consider a spring attached to a mass moving horizontally on a surface with friction that we rhythmically pull back and forth. After some initial transient period, the mass will move back and forth at the frequency of the force that we apply.

For a moment, let’s step back and consider the original undriven damped oscillator. We have two forces acting on the mass: the spring force \((-Kx)\) and the damping force \((-b\frac{dx}{dt})\). We now add a periodic driving force which can be written \(F_0\cos \omega t\). Newton’s second law now reads

\[
m \frac{d^2x}{dt^2} = -Kx - b \frac{dx}{dt} + F_0 \cos \omega t
\]  

Once a steady state has been reached, the motion will be sinusoidal with the same frequency as the driving force. The general form of the solution is the familiar

\[
x(t) = A \cos(\omega t + \phi)
\]  

The amplitude \(A\) and the phase \(\phi\) can be found by substituting (2) into (1) and requiring that the resulting equation be satisfied for all \(t\). Before proceeding, we rewrite (1) slightly. Dividing through by \(m\), we have

\[
\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{K}{m} x = \frac{F_0}{m} \cos \omega t
\]  

It is convenient to introduce the following notation, much of which is probably familiar by now.

\[
\frac{K}{m} = \omega_0^2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
This explains the subscript on $A_{LF}$; it refers to the amplitude of the motion at low frequencies.

If the driving frequency is not very low we have to find the amplitude $A$ and the phase $\phi$ of the motion by substituting (2) into (8). We will not go through the algebra here. The results are

\[
A = \frac{1}{\sqrt{(1 - r^2)^2 + (\frac{r}{Q})^2}} A_{LF}
\]

\[
\tan \phi = -\frac{1}{Q} \frac{r}{(1 - r^2)}
\]

in which $r = \omega/\omega_0$. $r$ is the ratio of the frequency of the driving force to the natural frequency of the oscillator. The factor on the right of (9) multiplying $A_{LF}$ is called the gain of the oscillator. The algebra further shows that $\phi$ is always in the third or fourth quadrant; the motion always lags behind the driving force.

We can say a few additional things about (9). First of all, for extremely low driving frequencies (compared to the natural frequency), $r$ is essentially zero, so that $A = A_{LF}$ independent of the value of $Q$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gain_curves.png}
\caption{Gain curves for $Q = 0.5, 1.0, 2.0, 4.0$, and $8.0$. Note that at $r = 1$, the curve peaks at $Q$.}
\end{figure}

To get a feel for the meaning of (9), look at Fig. 1. Notice that at large $Q$, i.e., for a small damping constant $b$, the curve is sharply peaked near $r = 1$. When the driving frequency is exactly equal to the natural frequency, $A = Q A_{LF}$. This steep increase in amplitude is called a “resonance”. The plot is $A/A_{LF}$ versus $r$ for $Q = 0.5, 1, 2, 4$, and $8$. There are many examples of resonance in physics and engineering.

You may wish to verify that the actual peak of the amplitude curve is reached not for $r = 1$ but for $r = \sqrt{1 - \frac{1}{4Q}}$. For large $Q$ (small damping) this makes very little difference; for for small values of $Q$ (large
damping) this shift becomes significant. For values of $Q$ smaller than $\sqrt{2}$, there is no maximum near the resonance at all. This can all be seen from Fig. 1.

For very high frequencies ($r \gg 1$), $A$ is very nearly $A_{LF}/r^2$ independent of $Q$. In this case, it is the acceleration term in (8) that predominates. We see then that both for very low frequencies and for very high frequencies the damping is relatively unimportant. It is mainly around resonance that the value of $Q$ determines the behavior of the oscillator.

Fig. 2 shows the phase of the oscillator in a graph with the same horizontal axis as Fig. 1 and for the same values of $Q$. Note that the phase is negative, i.e., the oscillator always lags behind the driving force. In the limit of large $Q$, the phase lag abruptly jumps from zero to $\pi$ as the driving frequency passes through $\omega_0$.

If these results were only applicable to a mass on a spring driven by a periodic force, they would be hardly worth studying. However, there is an enormous variety of problems that all lead to (1) except for the meaning of the constant parameters. Examples include a table on springs set in motion by a vibrating floor, the vibrations of an ammonia molecule driven by absorption of infrared radiation, and the charge on a capacitor in an LRC circuit driven by an AC voltage. It is this last case that we will investigate today.

The correspondence between mechanical and electrical parameters is shown in the following table:
### Mechanical System | Electrical System
---|---
Displacement $x$ | Charge $q$
Driving Force $F$ | Driving Voltage $V$
Mass $m$ | Inductance $L$
Damping Constant $b$ | Resistance $R$
Spring Constant $k$ | Reciprocal Capacitance $1/C$
Resonant Frequency $\sqrt{k/m}$ | Resonant Frequency $1/\sqrt{LC}$
Resonance Width $\gamma = b/m$ | Resonance Width $\gamma = R/L$

Using this correspondence, the governing equation becomes

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V_0 \cos \omega t$$

and all the other equations can be written in terms of the electrical parameters.

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**Sect. 2: Experiment.** The circuit will be constructed for you and connected to the function generator, which provides the driving voltage, and the oscilloscope. Calculate the resonant frequency for the components you have been given and the $Q$ of the circuit. Record the voltage across the capacitor and the delay of the peak of this signal relative to that of the driving signal as a function of frequency for a number of frequencies above and below the resonance. Take lots of data points where the gain curve changes rapidly near resonance. Note that you may have to continuously adjust the driving voltage to keep it constant as you change frequency. Make plots of your data similar to Fig. 1 and Fig. 2. How is the comparison?