Application of field-theoretic, collective-coordinate, and correlated-basis-function methods to many-boson systems*†

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(Received 18 February 1975)

The collective-coordinate and correlated-basis-function approaches are systematically applied to the determination of the elementary excitation spectrum $\omega(k)$ and the static structure function $S(k)$ for two models of a Bose gas which can be characterized by a single small expansion parameter g. Results are obtained to first order in g, the leading order in which multiexcitation processes are present, and are compared with those found by the microscopic-dielectric-function approach. It is found that a collective-coordinate calculation of $\omega(k)$ by Sunakawa, Yamasaki, and Kebukawa is incomplete to first order in g and that the convolution approximation for the three-particle distribution function in the correlated-basis-function formalism leads to incorrect results for overlap matrix element and for $S(k)$. A form for the overlap matrix element is proposed which leads to correct first-order results. Various properties of $\omega(k)$, $S(k)$, and the dynamic structure function are discussed.

I. INTRODUCTION

The correlated-basis function[1] and collective-coordinate[2] formalisms have been extensively applied to the calculation of the equilibrium properties of superfluid ^{4}He. Sunakava, Yamasaki, and Kebukawa[2] used a collective-coordinate approach to calculate the elementary excitation spectrum $\omega(k)$ and static structure $S(k)$ of superfluid ^{4}He and found qualitative agreement with experiment. Jackson and Feenberg[3] utilized the correlated-basis-function formalism and made the convolution approximation for the three-particle distribution function to obtain $\omega(k)$. However, some of the assumptions, e.g., the convolution approximation, made in the application of these formalisms lacks a firm theoretical basis. For this reason we apply in this paper the correlated-basis-function and collective-coordinate formalisms to two simple models of a zero-temperature Bose gas, which are characterized by a single small dimensionless parameter. Explicit numerical calculations to the first order beyond the Bogoliubov approximation are made for $\omega(k)$ and $S(k)$ and the results are compared with those found by using the microscopic-dielectric-function[4] approach.

We introduce in Sec. II the small parameter g and appropriate units for the dense charged Bose gas and the dilute Bose gas, respectively. The dielectric-function[4] formalism of Bose systems allows us to make rigorous perturbation approximations that are consistent with the general symmetry requirements of Bose systems. The first-order approximation is of interest since it repre-

sents the simplest approximation in which multi-excitation processes appear and is discussed in Sec. III A. We consider the first-order excitation spectrum $\omega^{(1)}(k)$, the dynamic structure function $S^{(1)}(k, \omega)$, and the static structure function $S^{(1)}(k)$ in Secs. III B–III D, respectively. Numerical results for $\omega^{(1)}(k)$ and $S^{(1)}(k)$ are shown for the charged Bose gas in Figs. 1 and 2. For the charged Bose gas it is found in particular that $\omega^{(1)}(k)$ is an analytic function of the wave vector k through at least order k^6, the high-frequency limit of $S^{(1)}(k, \omega)$ is proportional to $k^4\omega^{-11/2}$, and that due to the backflow and continuum contributions the single-excitation Feynman relation $S(k) = k^2/2m\omega(k)$ breaks down at order k^4.

We follow the general approach of Sunakawa, Yamasaki, and Kebukawa[2] (SYK) in Sec. IV A, express the Hamiltonian in terms of density and velocity operators, and retain terms in the Hamiltonian through first order in g. We use standard perturbation theory in Sec. IV B to obtain $\omega^{(1)}(k)$ and determine $S^{(1)}(k)$ in Sec. IV C using a method analogous to that used by Bhattacharyya and Woo[5] (BW). The results for $\omega^{(1)}(k)$ and $S^{(1)}(k)$ are shown to be consistent with the field-theoretic results of Sec. III, and an error in SYK's calculation of $\omega^{(1)}(k)$ is indicated. It is also seen that contrary to SYK's claim, it is possible to obtain $\omega^{(1)}(k)$ free of divergences using standard perturbation theory.

In Sec. V we discuss the general features of the correlated-basis-function calculation of BW in which $\omega^{(1)}(k)$ is determined by two independent methods. The first method outlined in Sec. V A is based on the Bijl-Dingle-Jastrow form[1] of the trial

function and yields a $\omega^{(1)}(k)$ that is consistent with the results of Secs. III B and IV B. In the method given in Sec. V B the ground-state correlations are assumed to be known exactly. The convolution approximation[1] for the three-particle distribution function is made in order to evaluate a certain matrix element (usually referred to as the overlap integral) which enters into $\omega^{(1)}(k)$. The second method yields an expression for $\omega^{(1)}$ in terms of $S^{(1)}$, and $S^{(1)}$ is determined indirectly by setting equal the two expressions for $\omega^{(1)}(k)$. The result for $S^{(1)}(k)$ is shown in Fig. 2. We summarize and discuss our results in Sec. VI. A brief report of some of our results was given in Ref. 6.

II. SMALL PARAMETERS

We consider a system of N spinless bosons at zero temperature of mass m enclosed in a volume V with the density $n = N/V$ a fixed constant. The Hamiltonian is

$$H = \sum_k \epsilon_k a_k^\dagger a_k + \frac{1}{2V} \sum_{k, k', q} v(q) a_{k+q}^\dagger a_{k'}^\dagger a_{k'+q} a_k, \quad (2.1)$$

where $\epsilon_k = k^2/2m$,

$$v(q) = \int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}} v(r)$$

is the Fourier transform of the interparticle potential $v(r)$, and $a_k^\dagger$ and a_k are the usual boson creation and destruction operators. We choose units such that $\hbar = 1$ and $V = 1$. The procedure[4] for determining a well-defined field-theoretic model in the presence of a macroscopically occupied zero-momentum state is well known.

We consider two models of a weakly interacting Bose gas which can be characterized by a single small dimensionless parameter. For the high-density charged Bose gas we shall use the system of units in which the energy is measured in the plasma frequency $\omega_p = (4\pi e^2 n/m)^{1/2}$ and the momentum is measured in terms of $(\omega_p m)^{1/2}$. The small dimensionless parameter g is defined as

$$g = (4\pi e^2 m)^{3/4} n^{-1/4}. \quad (2.2)$$

In the new system of units we have

$$\omega_p = m = 1, \quad g = n^{-1} = 4\pi e^2. \quad (2.3)$$

We also consider a simple model of a Bose gas for which $v(k)$ can be summarized by the s-wave scattering length a. We use the system of units in which the energy is measured in terms of ms_0^2 and momentum in terms of ms_0, where $s_0 = (4\pi an)^{1/2}/m$ is the Bogoliubov sound speed. The parameter g is defined as

$$g = 4\pi am s_0. \quad (2.4)$$

We now have a new set of units in which

$$s_0 = m = 1, \quad g = n^{-1} = 4\pi a. \quad (2.5)$$

To the order of approximation of interest the parameter v can be eliminated by using the perturbation expansion for the scattering length a in powers of v. We write

$$g = 4na = v + \int \frac{d^3p}{(2\pi)^3} v \frac{1}{-2\epsilon_p} v + O(v^3), \quad (2.6)$$

or rearranging (2.6) we find,

$$v/g = 1 + gv^{(1)} + O(g^2), \quad (2.7)$$

where

$$v^{(1)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\epsilon_p}. \quad (2.8)$$

The large-p divergence in (2.8) arises from the point nature of our model interaction. It will cancel other divergent terms so that the physical quantities of interest are well defined. It will be convenient in the following to define the quantity

$$\phi(k) = \begin{cases} 1/k^2, & \text{charged bose gas} \\ 1, & \text{dilute bose gas.} \end{cases} \quad (2.9)$$

III. DIELECTRIC FORMULATION

A. Formalism

The generalized dielectric formulation[4,7] of Bose systems allows us to make rigorous perturbation approximations that explicitly show the coincidence of the single-particle and denisty spectrums, satisfy local number conservation, and yield a gapless excitation spectrum. The dielectric formulation has been used by Ma and Woo[7] to obtain the first-order correction to the excitation spectrum of the charged Bose gas and by Wong and Gould[4] to obtain the first-order correction to the excitation spectrum, dynamic structure function, and static structure function of a dilute Bose gas. Since the dielectric approach has been reviewed in detail in Ref. 4, we collect here only the important formulas.

The dynamical variables of interest are the amplitude operator

$$a_{k\mu} = \begin{cases} a_k, & \mu = + \\ a_{-k}^\dagger, & \mu = -. \end{cases} \quad (3.1)$$

The density operator

$$\rho(k) = n_0^{1/2} a_{k\mu} \delta_\mu + \sum_p{}' a_p^\dagger a_{p+k}, \quad (3.2)$$

and the current operator

$$\vec{J}(k) = (\tfrac{1}{2}\vec{k}) n_0^{1/2} a_{k\mu} \delta_\mu + \sum_p{}' (\vec{p} + \tfrac{1}{2}\vec{k}) a_p a_{p+k}, \quad (3.3)$$

where $\delta_\mu = 1$, $\beta_\mu = \text{sgn}\mu$, and the prime on the summation sign indicates zero momentum operators are excluded. Repeated Greek indices are summed.

We define the (matrix) amplitude, density, and current response functions, respectively,

$$g_{\mu\nu}(k,\omega) = -i\int dt\, e^{i\omega t}\langle[a_{k\mu}(t), a^\dagger_{k\nu}]\rangle\Theta(t),\quad (3.4)$$

$$\mathcal{F}(k,\omega) = -i\int dt\, e^{i\omega t}\langle[\rho_k(t),\rho^\dagger_k]\rangle\Theta(t),\quad (3.5)$$

$$\mathcal{F}^{zz}(k,\omega) = -i\int dt\, e^{i\omega t}\langle[J^z_k(t), J^{z\dagger}_k]\rangle\Theta(t),\quad (3.6)$$

where the superscript z denotes the component of $\vec{J}$ in the direction of $\hat{k}$.

In the dielectric-function approach the response functions are expressed in terms of the corresponding regular functions (those that do not involve an isolated single-interaction nor an isolated one-particle line). In this way the denominators of the response functions (3.4)–(3.6) can be related to the dielectric function $\epsilon(k,\omega)$ which is defined by

$$\mathcal{F} = F/\epsilon,\quad (3.7a)$$

$$\epsilon = 1 - vF,\quad (3.7b)$$

and, in particular, F^{zz} can be written as

$$F^{zz} = \Lambda^z_\mu G_{\mu\nu}\Lambda^z_\nu + F^{zzr}.\quad (3.8)$$

The irreducible functions (those that contain no isolated single-interaction line) of g, $\mathcal{F}$ and $\mathcal{F}^{zz}$ are denoted by capital letters G, F, and F^{zz}. Regular functions are further distinguished if necessary by a superscript r. The (regular) vertex function Λ^z_μ joins the one-particle line to the longitudinal current vertex. An application of the continuity equation yields the relation

$$\omega^2 F = k^2(F^{zz} + n/m).\quad (3.9)$$

We see from (3.7)–(3.9) that an approximation for the regular longitudinal current response functions F^{zzr} and Λ^z_μ automatically leads to approximate density response function that is consistent with local number conservation.

We develop a perturbation expansion and write for example

$$g^{1/2}\Lambda^z_\mu = \tfrac{1}{2}k\beta_\mu(1 - \tfrac{1}{2}gn'^{(1)}) + g\Lambda^{z(1)}_\mu + O(g^2),\quad (3.9a)$$

$$gn^1 = gn'^{(1)} + O(g^2),\quad (3.9b)$$

$$G_{\mu\nu} = G^{(0)}_{\mu\nu} + gG^{(1)}_{\mu\nu} + O(g^2),\quad (3.9c)$$

$$gF^{zzr} = gF^{zzr(1)} + O(g^2),\quad (3.9d)$$

$$gF^{zz} = F^{zz(0)} + gF^{zz(1)} + O(g^2),\quad (3.9e)$$

and collect the $O(1)$ and $O(g)$ terms in (2.7) and (3.7)–(3.9). We use the fact that $F^{zz(0)} = \epsilon^2_k(\omega^2 - \epsilon^2_k)^{-1}$ and find after some algebra

$$\mathcal{F}^{(0)} = k^2[\omega^2 - \omega^{(0)}(k)^2]^{-1}\quad (3.10)$$

and

$$\mathcal{F}^{(1)}(k,\omega) = (k^2/\omega^2)[\omega^2 - \omega^{(0)}(k)^2]^{-2}\mathfrak{N}^{zz(1)}(k,\omega),$$
$$(3.11)$$

where

$$\mathfrak{N}^{zz(1)}(k,\omega) = \epsilon_k(\omega^2 - \epsilon^2_k)^2[S^{(1)} + M^{(1)}_2 - \mu^{(1)}$$
$$+ 2\omega\delta_\mu\Lambda^{z(1)}_\mu/k + k\beta_\mu\Lambda^{z(1)}_\mu - \epsilon_k n'^{(1)}]$$
$$+ (\omega^2 - \epsilon^2_k)^2 F^{zzr(1)}$$
$$+ k^2[\omega\epsilon_k A^{(1)} + \epsilon^2_k(S^{(1)} - \mu^{(1)}) + \omega^2 v^{(1)}].$$
$$(3.12)$$

In (3.12) $S = \tfrac{1}{2}(M_{++} + M_{--})$, $A = \tfrac{1}{2}(M_{++} - M_{--})$, $M_2 = M_{+-}$, and $M_{\mu\nu}$ is the irreducible self-energy. To avoid confusion we note that the term involving $v^{(1)}$ in (3.12) only enters for the dilute Bose gas. The $O(1)$ Bogoliubov excitation spectrum $\omega^{(0)}_k$ is

$$\omega^{(0)}(k) = k[\tfrac{1}{4}k^2 + \phi(k)]^{1/2}.\quad (3.13)$$

The one-loop diagrams for $M^{(1)}$, $\Lambda^{z(1)}_\mu$, and $F^{zzr(1)}$ are shown in Fig. 6 of Ref. 4. The evaluation of those diagrams is straightforward and the corresponding one-loop integrals are

$$S^{(1)} + M^{(1)}_2 - \mu^{(1)} = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\phi(p)[\lambda_{p+k} - \lambda_p]$$
$$+ \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\phi(p)\lambda_p\lambda_{p+k}$$
$$\times[\phi(p) + \phi(\vec{p}+\vec{k})]Q^+,\quad (3.14)$$

$$S^{(1)} - M^{(1)}_2 - \mu^{(1)} = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\phi(p)(1/\lambda_{p+k} - \lambda_p)$$
$$+ \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\phi(p)$$
$$\times[\phi(\vec{p}+\vec{k}) + \phi(p)\lambda_p/\lambda_{p+k}]Q^+,\quad (3.15)$$

$$A^{(1)} = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\phi(p)\lambda_p[\phi(p) + \phi(\vec{p}+\vec{k})]Q^-,\quad (3.16)$$

$$\delta_\mu\Lambda^{z(1)}_\mu = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\phi(p)(\lambda_p - \lambda_{p+k})(\vec{p}\cdot\hat{k} + \tfrac{1}{2}k)Q^-,\quad (3.17)$$

$$\beta_\mu\Lambda^{z(1)}_\mu = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\phi(p)(\lambda_p/\lambda_{p+k} - 1)(\vec{p}\cdot\hat{k} + \tfrac{1}{2}k)Q^+,$$
$$(3.18)$$

$$F^{zzr(1)} = \frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\frac{(1 - \lambda_p)(\lambda_{p+k} - \lambda_p)}{\lambda_p}(\vec{p}\cdot\hat{k} + \tfrac{1}{2}k)^2 Q^+,$$
$$(3.19)$$

$$n'^{(1)} = \frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\frac{(1 - \lambda_p)^2}{\lambda_p},\quad (3.20)$$

with

$$Q^\pm = [\omega - \omega^{(0)}(\vec{p}+\vec{k}) - \omega^{(0)}(p)]^{-1}$$
$$\mp[\omega + \omega^{(0)}(\vec{p}+\vec{k}) + \omega^{(0)}(p)]^{-1}\quad (3.21)$$

and

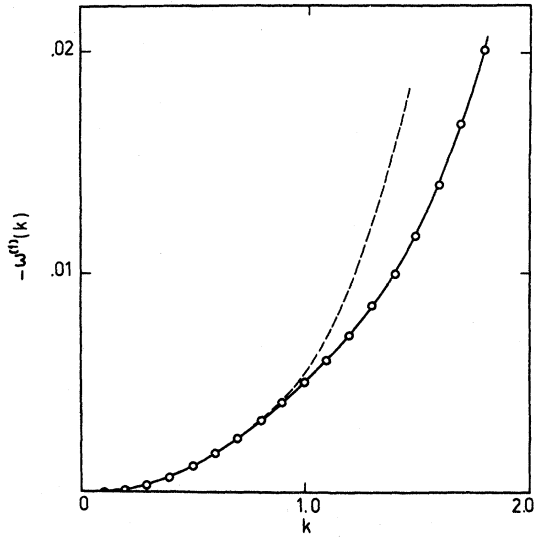$$\lambda_p = \epsilon_p/\omega^{(0)}(p).\quad (3.22)$$

FIG. 1. First-order excitation spectrum of the charged Bose gas. The solid line is the result of the dielectric formulation. The open circles are the result of the collective-coordinate and the correlated-basis-function calculations. The dotted line represents the long-wavelength expansion of $\omega^{(1)}(k)$. Energy is measured in terms of the plasma frequency ω_p, and k is measured in units of $(\omega_p m)^{1/2}$.

B. Elementary excitation spectrum

The elementary excitation spectrum $\omega(k)$ is determined by the zero of $\epsilon(k, \omega)$. We use (3.7b) and (3.9) to rewrite the equation $\epsilon = 0$ as

$$\omega(k)^2/k^2 = (v/g)[1 + gF^{zz}(k, \omega)] , \tag{3.23}$$

and develop a perturbation expansion for $\omega(k)$

$$\omega(k) = \omega^{(0)}(k) + g\omega^{(1)}(k) + O(g^2) , \tag{3.24}$$

$$gF^{zz} = F^{zz(0)} + g\left(F^{zz(1)} + 2\omega^{(0)}(k)\omega^{(1)}(k)\frac{\partial F^{zz(0)}}{\partial \omega^2}\right)$$
$$+ O(g^2) . \tag{3.25}$$

All quantities on the right-hand side of (3.25) are evaluated at $\omega^{(0)}(k)$. If we use the form of $F^{zz(0)}$ given above and collect the $O(1)$ terms in (3.23)–(3.25), we find the result (3.13) for $\omega^{(0)}(k)$. To $O(g)$ we find

$$\omega^{(1)}(k) = \mathfrak{R}^{zz(1)}(k, \omega^{(0)}(k))/2\omega^{(0)}(k)^3 . \tag{3.26}$$

We consider the evaluation of $\omega^{(1)}(k)$ for the two models of Sec. II. For the charged Bose gas $\phi(k) = 1/k^2$, and we write (3.13) and (3.22) as

$$\omega^{(0)}(k) = (1 + \tfrac{1}{4}k^4)^{1/2} , \tag{3.27}$$

$$\lambda_k = k^2/2(1 + \tfrac{1}{4}k^4)^{1/2} . \tag{3.28}$$

We find that in the long-wavelength limit $\omega^{(1)}(k)$ has the form

$$\omega^{(1)}(k) = c_2 k^2 + c_4 k^4 + c_5 k^5 + O(k^6) , \tag{3.29}$$

where

$$c_2 = -\frac{1}{32\pi^2}\int_0^\infty dp\, \frac{p^2(p^4 + \tfrac{68}{15})}{(p^4 + 3)(\tfrac{1}{4}p^4 + 1)^{3/2}}$$

$$= -4.679 \times 10^{-3} , \tag{3.30}$$

$$c_4 = 1.881 \times 10^{-3} , \tag{3.31}$$

$$c_5 = -\frac{1}{192\pi^2}\int_0^\infty dp\, \frac{1}{p}\ln\left|\frac{p+1}{p-1}\right|$$

$$= -2.604 \times 10^{-3} . \tag{3.32}$$

The coefficients c_2 and c_4 have been evaluated numerically. The integral for c_4 is given in Ref. 8. Note that $\omega^{(1)}(k)$ in (3.29) has both even and odd powers of k and that the k^5 term arises from the expansion of (3.15). The coefficient c_2 was first obtained in Ref. 7; the coefficients are believed to be new. The result of a numerical evaluation of (3.26) for $\phi = 1/k^2$ is shown in Fig. 1 and is in agreement with Ref. 7. We also show for comparison the small-k expansion (3.29).

For the dilute Bose gas $\phi = 1$, we write (3.13) and (3.22) as

$$\omega^{(0)}(k) = k(1 + \tfrac{1}{4}k^4)^{1/2} , \tag{3.33}$$

$$\lambda_k = k/2(1 + \tfrac{1}{4}k^4)^{1/2} . \tag{3.34}$$

The small-k expansion of $\omega^{(1)}(k)$ for $\phi = 1$ is given in Ref. 4 in which it is shown that $\omega^{(1)}(k)$ is a nonanalytic function of k in contrast to (3.29). We do not evaluate $\omega^{(1)}(k)$ numerically for $\phi = 1$ because of the appearance of vanishing denominators in the integrals (3.14)–(3.19) which make the resulting principal-value integrals not really amenable to numerical evaluation.

C. Dynamic structure function

At $T = 0$, the dynamic structure function $S(k, \omega)$ is the positive frequency spectral function of $\mathfrak{F}(k, \omega)$

$$S(k, \omega) = \begin{cases} -(1/\pi)\,\mathrm{Im}\mathfrak{F}(k, \omega), & \omega \geq 0 \\ 0, & \omega < 0 . \end{cases} \tag{3.35}$$

To $O(g)$, $S(k, \omega)$ is found from (3.10) and (3.11) to have the form

$$gS(k, \omega) = Z_F^{(0)}(k)\delta(\omega - \omega^{(0)}(k) - g\omega^{(1)}(k))$$
$$+ gZ_F^{(1)}(k)\delta(\omega - \omega^{(0)}(k))$$
$$+ gY^{(1)}(k)\delta(\omega - \omega^{(0)}(k)) + gX^{(1)}(k, \omega) , \tag{3.36}$$

where

$$Y^{(1)}(k, \omega) = -[k^2/4\omega^{(0)}(k)^2]$$
$$\times \frac{d}{d\omega}\left[\omega^{-2}\mathfrak{R}^{zz(1)}(k, \omega)\right]_{\omega^{(0)}(k)} , \tag{3.37}$$

$$X^{(1)}(k, \omega) = -k^2\omega^{-2}\,\mathrm{Re}[\omega^2 - \omega^{(0)}(k)^2]^{-2}$$
$$\times (1/\pi)\,\mathrm{Im}\mathfrak{R}^{zz(1)}(k, \omega) , \tag{3.38}$$

with

$$Z_F^{(0)}(k) = k^2/2m\omega^{(0)}(k)$$

and

$$Z_F^{(1)}(k) = -Z_F^{(0)}(k)\omega^{(1)}(k)/\omega^{(0)}(k).$$

As discussed by Wong and Gould[4] the first two terms on the right-hand side of (3.36) represent the single-excitation contribution which exhausts the f-sum rule, the backflow term $Y^{(1)}(k)$ arises from virtual multiexcitation processes, and the continuum $X^{(1)}(k, \omega)$ represents the contribution of real multiexcitations. It is easy to show that (3.36) satisfies all the relevant sum rules.

It is of interest to determine the high-frequency behavior of $S(k, \omega)$ where only (3.38) contributes. In the limit $\omega \gg \omega^{1/2}k \gg 1$, we can write ($\omega > 0$)

$$-(1/\pi)\,\mathrm{Im}Q^\pm = \delta(\omega - p^2),\qquad(3.39)$$

and $X^{(1)}(k, \omega)$ takes the form

$$X^{(1)}(k, \omega) = (k^4/8\pi^2\omega^{7/2})\phi(\omega^{1/2})^2\left(\frac{7}{15}+\frac{\omega}{2}\int du\,u^2\phi_1^2\right.$$
$$\left. +\omega^{1/2}\int du(u\phi_1 - 2u^3\phi_1)\right),\quad(3.40)$$

where ϕ_1 is defined by $\phi_{p+k}/\phi_p = 1 + \phi_1 k + \cdots$ with $p = \omega^{1/2}$ and $u = \cos(\vec{k}, \vec{p})$. For the charged Bose gas $\phi_1 = -2u/\omega^{1/2}$ and from (3.40)

$$X^{(1)}(k, \omega) = (23/120\pi^2)k^4\omega^{-11/2}.\qquad(3.41)$$

For a dilute Bose gas $\phi_1 = 0$ and

$$X^{(1)}(k, \omega) = (7/120\pi^2)k^4\omega^{-7/2},\qquad(3.42)$$

as was obtained in Ref. 4, in which the interpretation of (3.42) was discussed. It is interesting to also consider a potential of the form $\phi(p) = e^{-p^2\alpha^2}$ for which we find from (3.40) ($\phi_1 = -2\alpha^2\omega^{1/2}u$)

$$X^{(1)}(k, \omega) = (7/120\pi^2)k^4\omega^{-7/2}e^{-\omega\alpha^2}.\qquad(3.43)$$

An inspection of (3.43) shows that there is an intermediate frequency range in which $X^{(1)}(k, \omega)$ has the same frequency dependence as (3.42).

D. Static structure function

The static structure function is given by

$$S(k) = \int_0^\infty d\omega\,S(k, \omega).\qquad(3.44)$$

We write

$$gS(k) = S^{(0)}(k) + gS^{(1)}(k) + O(g^2),\qquad(3.45)$$

and find from (3.36) and (3.22)

$$S^{(0)}(k) = Z_F^{(0)}(k) \equiv \lambda_k\qquad(3.46)$$

and

$$S^{(1)}(k) = Z_F^{(1)}(k) + Y^{(1)}(k) + X^{(1)}(k),\qquad(3.47)$$

with

$$X^{(1)}(k) = \int_0^\infty d\omega\,X^{(1)}(k, \omega).$$

We see that $Y^{(1)}$ and $X^{(1)}$ are the $O(g)$ corrections to the single-excitation Feynman relation $S(k) = k^2/2\omega(k)$.

For the charged Bose gas it is easy to find from (3.29), (3.37), and (3.38) the small-k behavior $Z_F^{(1)}(k) = 2.339 \times 10^{-3}k^4$, $Y^{(1)}(k) = -1.085 \times 10^{-3}k^4$, $X^{(1)}(k) = 4.57 \times 10^{-4}k^4$, and

$$S^{(1)}(k) = s_4 k^4 + O(k^6).\qquad(3.47a)$$

where

$$s_4 = \frac{1}{240\pi^2}\int_0^\infty \frac{dp\,p^2[15\omega^{(0)}(p)^2 + 15\omega^{(0)}(p) - 2]}{\omega^{(0)}(p)^3[2\omega^{(0)}(p) + 1]^2}$$
$$= -1.711 \times 10^{-3}.\qquad(3.47b)$$

The integrals leading to (3.47) have been evaluated numerically. Both $Y^{(1)}(k)$ and $X^{(1)}(k)$ are $O(k^4)$, and the Feynman relation breaks down at $O(k^4)$. A numerical evaluation of $S^{(1)}(k)$ is shown in Fig. 2.

IV. METHOD OF COLLECTIVE COORDINATES

A. Hamiltonian in terms of collective coordinates

The Hamiltonian (2.1) can be expressed in terms of collective variables in a variety of ways. Bogoliubov and Zubarev[9] introduced the density operator as a collective variable in a microscopic formulation of many-boson systems. However, in such a formulation,[10] special techniques must be developed to treat the non-Hermicity of the resulting Hamiltonian. Current and density operators have been considered by several workers[11] as a pair of dynamical variables. Since the current and density operators do not form a canonically conjugate pair of variables, this approach is not convenient. One of the more successful approaches has been that
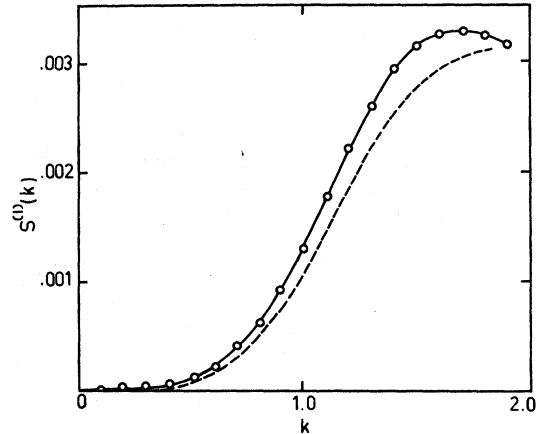

FIG. 2. First-order static structure function of the charged Bose gas. The solid line is the result of the dielectric formulation. The open circles are the result of the collective-coordinate approach. The dashed line is the correlated-basis-function result of Bhattacharyya and Woo. k is measured in $(\omega_p m)^{1/2}$.

of Sunakawa, Yamasaki, and Kebukawa[2] (SYK) who introduced a canonically conjugate density operator and velocity operator. We follow their general approach and define the density operator $\rho(k)$ and the velocity operator $\vec{u}(k)$ as

$$\rho(k) = N^{-1/2} \sum_p a_p^\dagger a_{p+k}, \qquad (4.1)$$

$$\vec{u}(k) = \vec{J}(k) - N^{-1/2} \sum_{p \neq k} \rho(\vec{p} - \vec{k})\vec{u}(p), \qquad (4.2)$$

where

$$\vec{J}(k) = N^{-1/2} \sum_p (\vec{p} + \tfrac{1}{2}\vec{k})a_p^\dagger a_{p+k}. \qquad (4.3)$$

We have introduced the factor of $N^{-1/2}$ in the definition of ρ and $\vec{J}$ so that the N dependence of the various terms of the Hamiltonian is transparent and we shall use the new normalization from hereon. The collective-coordinate approach is based on the assumption that the density fluctuations are small compared with the average density, i.e.,

$$\rho'(x) = \rho(x) - n \ll n, \qquad (4.4)$$

so that we can write

$$1/\rho(x) = (1/n)\{1 - \rho'(x)/n + [\rho'(x)/n]^2 - \cdots\}. \qquad (4.5)$$

We restrict the domain of the velocity operator to the subspace of the Hilbert space defined by $\vec{k} \times \vec{u}(k) = 0$. In this subspace the $\vec{u}(k)$ commute among themselves. The Hamiltonian can thus be written in the form

$$H = H_0 + H_1 + H_2 + \cdots, \qquad (4.6)$$

where

$$H_0 = \frac{N(N-1)}{2} v(0) - \sum_k{}' \left[\tfrac{1}{4}k^2 + \tfrac{1}{2}nv(k)\right]$$
$$+ \frac{1}{2}\sum_k{}' \{[\tfrac{1}{4}k^2 + nv(k)]\rho(k)\rho(-k) + \vec{u}(k) \cdot \vec{u}(-k)\}, \qquad (4.7)$$

$$H_1 = \frac{1}{2\sqrt{N}} \sum_{\substack{k \neq -p \\ k, p}}{}' \left[\vec{u}(k) \circ \rho(\vec{k} + \vec{p}) \cdot \vec{u}(p)\right.$$
$$+ \left.\tfrac{1}{4}(\vec{k} \cdot \vec{p})\rho(k)\rho(-\vec{k} - \vec{p})\rho(p)\right], \qquad (4.8)$$

and

$$H_2 = \frac{1}{8N} \sum_{\substack{k+p+q \neq 0 \\ k, p, q}}{}' \vec{k} \cdot (\vec{p} + \vec{q} + \vec{k})\rho(p)\rho(q)\rho(k)$$
$$\times \rho(-\vec{q} - \vec{k} - \vec{p}). \qquad (4.9)$$

The terms H_0, H_1, and H_2 contain products of two, three, and four operators, respectively. In the following we shall write $g = 1/N$ in H_1 and H_2, because $V = 1$. Note that we have retained terms to $O(g)$ in H.

The Hamiltonian (4.6) can be simplified by diagonalizing the quadratic part of H_0. We write

$$\rho(k) = \lambda_k^{1/2}[c(k) + c^\dagger(-k)], \qquad (4.10)$$

$$\vec{u}(k) = \tfrac{1}{2}\vec{k}\lambda_k^{-1/2}[c(-k) - c^\dagger(k)]. \qquad (4.11)$$

The operators $c^\dagger(k)$ and $c(k)$ satisfy the usual Bose commutation relations. We use (4.10) and (4.11) to write (4.7) as

$$H_0 = E_0 + \sum_k \omega_0(k)c^\dagger(k)c(k), \qquad (4.12)$$

with

$$E_0 = \tfrac{1}{2}N(N-1)v(0)$$
$$+ \frac{1}{2}\sum_k{}' \left[\omega_0(k) - \tfrac{1}{2}k^2 - nv(k)\right]. \qquad (4.13)$$

In the above, λ_k is given by (3.46) and $\omega_0(k)$ is defined as $\omega_0(k) = k(\tfrac{1}{4}k^2 + v/g)^{1/2}$. For the charged Bose gas $v/g = \phi$ and $\omega_0(k)$ is identical to $\omega^{(0)}(k)$ given in (3.27). However, for the dilute Bose gas we have from (2.7)

$$\omega_0(k) = \omega^{(0)}(k) + gk^2 v^{(1)}/2\omega^{(0)}(k), \qquad (4.14)$$

where $\omega^{(0)}(k)$ is given in (3.33).

H_1 and H_2 can be written in terms of the excitation operators c and $c^\dagger$ as

$$H_1 = g^{1/2} \sum_{k, p; k \neq -p} A_1(k, p)[c(k)c(-\vec{k} - \vec{p})c(p) + c^\dagger(\vec{k} + \vec{p})c(k)c(p) + \text{H. c.}]$$
$$+ g^{1/2} \sum_{k, p; k \neq -p} A_2(k, p) [c^\dagger(k)c(\vec{k} + \vec{p})c(-p) + \text{H. c.}] \qquad (4.15)$$

and

$$H_2 = g \sum_{k; k+p+q \neq 0}{}' A_3(k, p, q) \{[c(p) + c^\dagger(-p)][c(q) + c^\dagger(-q)][c(k) + c^\dagger(-k)][c(-\vec{p} - \vec{q} - \vec{k}) + c^\dagger(\vec{p} + \vec{q} + \vec{k})]\}, \qquad (4.16)$$

where

$$A_1(k, p) = \tfrac{1}{8}(\vec{k} \cdot \vec{p})\left(\frac{\lambda_{k+p}}{\lambda_k \lambda_p}\right)^{1/2}(1 + \lambda_k \lambda_p), \qquad (4.17)$$

$$A_2(k, p) = -\tfrac{1}{4}(\vec{k} \circ \vec{p})\left(\frac{\lambda_{k+p}}{\lambda_k \lambda_p}\right)^{1/2}(1 - \lambda_k \lambda_p), \qquad (4.18)$$

and

$$A_3(k, p, q) = \tfrac{1}{8}(\lambda_k \lambda_p \lambda_q \lambda_{p+q+k})^{1/2} [\vec{k} \cdot (\vec{k} + \vec{p} + \vec{q})]. \quad (4.19)$$

B. Elementary excitation spectrum

We use standard perturbation theory to evaluate the excitation spectrum to $O(g)$. The ground state of the system corresponds to the vacuum state $|0\rangle$. The multiexcitation states are given by

$$|\vec{n}\rangle = |\vec{k}_1, \vec{k}_2, \vec{k}_3, \ldots, \vec{k}_n\rangle$$

$$= \prod_{i=1} c^\dagger(\vec{k}_i)|0\rangle, \quad (4.20)$$

with $\sum_i k_i = \vec{k}$ and $\Pi_i k_i \neq 0$, $n = 1, 2, 3, \ldots$.

The excitation spectrum can be found from the difference between the energy of the first excited state and the ground-state energy. In second-order perturbation theory we write $\omega^{(1)}(k)$ as

$$\omega^{(1)}(k) = \omega_1(k) + \omega_2(k), \quad (4.21)$$

with

$$\omega_1(k) = \langle k|H'|k\rangle - \langle 0|H'|0\rangle, \quad (4.22)$$

$$\omega_2(k) = \sum_{n \neq k} \frac{|\langle n|H'|k\rangle|^2}{\langle k|H_0|k\rangle - \langle n|H_0|n\rangle}$$

$$- \sum_{m \neq 0} \frac{|\langle m|H'|0\rangle|^2}{\langle 0|H_0|0\rangle - \langle m|H_0|m\rangle}, \quad (4.23)$$

where $H' = H_1 + H_2$. Note that there is an additional term $k^2 v^{(1)}/2\omega^{(0)}(k)$ in (4.21) for the dilute Bose gas. Since H_1 is a product of three operators, its expectation value in (4.22) vanishes. If we substitute (4.16) into (4.22) we find

$$\omega_1(k) = \frac{\lambda_k}{4} \int \frac{d^3p}{(2\pi)^3} (\vec{k} + \vec{p})^2 \lambda_p. \quad (4.24)$$

We use (4.20) to write (4.23) in the form

$$\omega_2(k) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{|\langle \vec{k} - \vec{p}, \vec{p}|H_1|\vec{k}\rangle|^2}{\langle \vec{k}|H_0|\vec{k}\rangle - \langle \vec{k} - \vec{p}, \vec{p}|H_0|\vec{p}, \vec{k} - \vec{p}\rangle} + \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{|\langle \vec{k}, \vec{p}, -\vec{k} - \vec{p}|H_1|0\rangle|^2}{\langle 0|H_0|0\rangle - \langle \vec{k}, \vec{p}, -\vec{k} - \vec{p}|H_0|-\vec{k} - \vec{p}, \vec{p}, \vec{k}\rangle}, \quad (4.25)$$

where

$$\langle \vec{k}|H_0|\vec{k}\rangle = E_0 + \omega_0(k),$$

$$\langle \vec{k} - \vec{p}, \vec{p}|H_0|\vec{p}, \vec{k} - \vec{p}\rangle = E_0 + \omega_0(p) + \omega_0(|\vec{k} - \vec{p}|), \quad (4.26)$$

$$\langle \vec{k}, \vec{p}, \vec{q} - \vec{k} - \vec{p}|H_0|\vec{q} - \vec{k} - \vec{p}, \vec{p}, \vec{k}\rangle = E_0 + \omega_0(k) + \omega_0(p) + \omega_0(|\vec{q} - \vec{k} - \vec{p}|).$$

It is straightforward to use (4.15), (4.17), and (4.18) to write the matrix elements in (4.25) as

$$\langle \vec{k} - \vec{p}, \vec{p}|H_1|\vec{k}\rangle = \tfrac{1}{4}(\lambda_k \lambda_p \lambda_{|\vec{k} - \vec{p}|})^{-1/2}[(k^2 + p^2 - \vec{k} \cdot \vec{p})\lambda_k \lambda_p \lambda_{|\vec{k} - \vec{p}|} - \vec{p} \cdot (\vec{k} - \vec{p})\lambda_k - \vec{p} \cdot \vec{k}\lambda_{|\vec{k} - \vec{p}|} - (\vec{k} - \vec{p}) \cdot \vec{k}\lambda_p], \quad (4.27)$$

$$\langle \vec{k}, \vec{p}, -\vec{k} - \vec{p}|H_1|0\rangle = \tfrac{1}{4}(\lambda_k \lambda_p \lambda_{|\vec{k} + \vec{p}|})[-(k^2 + p^2 + \vec{k} \cdot \vec{p})\lambda_k \lambda_p \lambda_{|\vec{k} + \vec{p}|} - \vec{p} \cdot (\vec{k} + \vec{p})\lambda_k + \vec{p} \cdot \vec{k}\lambda_{|\vec{k} + \vec{p}|} + (\vec{k} + \vec{p}) \cdot \vec{k}\lambda_p], \quad (4.28)$$

and to substitute (4.26)–(4.28) into (4.25) to obtain the integral for $\omega_2(k)$. The integral (4.24) for $\omega_1(k)$ is divergent for large p but this divergence is canceled exactly by a similar divergence in the integral (4.25) for $\omega_2(k)$ [and by the term proportional to $v^{(1)}$ in (4.14) for the dilute Bose gas]. Thus we see that it is possible to use standard perturbation theory to obtain an excitation spectrum to first order in g free of divergences in contrast to SYK's claim that standard perturbation theory leads to divergent results.

The explicit appearance of the potential ϕ in the integrals (3.14)–(3.18) makes it difficult to show that the above result for $\omega^{(1)}(k)$ agrees analytically for all k with the field-theoretic result (3.26). Therefore, we limit ourselves to showing that the two results for $\omega^{(1)}(k)$ agree analytically in the limit $k \to 0$. For the choice $\phi = 1/k^2$ we can use (3.27) and write (4.24) in the limit $k \to 0$ as

$$\omega_1(k) = \frac{k^2}{32\pi^2} \int_0^\infty dp\, p^6(1 + \tfrac{1}{4}p^4)^{-1/2}. \quad (4.29)$$

In the same manner we can write the matrix elements (4.27) and (4.28) for $k \to 0$ as

$$\langle \vec{p}, \vec{k} - \vec{p}|H_1|\vec{k}\rangle = k(32)^{-1/2}[p^2\lambda_p + p^2\lambda_p^{-1} - 2 + 8(\hat{k} \cdot \vec{p})^2\lambda_p p^{-2}], \quad (4.30)$$

$$\langle \vec{k}, \vec{p}, -\vec{k} - \vec{p}|H_1|0\rangle = \langle \vec{p}, \vec{k} - \vec{p}|H_1|\vec{k}\rangle - k(32)^{-1/2}[2p^2(\lambda_p + \lambda_p^{-1})]. \quad (4.31)$$

If we substitute (4.30) and (4.31) in (4.25) and take the $k \to 0$ limit of (4.26) we find

$$\omega_2(k) = -\frac{k^2}{128\pi^2} \int_0^\infty dp\, \frac{p^2(p^{12} + 7p^8 + 16p^4 + \tfrac{172}{15})}{(p^4 + 3)(1 + \tfrac{1}{4}p^4)^{3/2}}, \quad (4.32)$$

so that from (4.21), (4.29), and (4.32) we have

$$\omega^{(1)}(k) = - \frac{k^2}{32\pi^2} \int_0^\infty dp \, \frac{p^2(p^4 + \frac{68}{15})}{(p^4+3)(1+\frac{1}{4}p^4)^{3/2}} \, . \quad (4.33)$$

We see that (4.33) is identical to (3.30). We have numerically evaluated the integrals (4.24) and (4.25) for the charged Bose gas up to $k = 2$, and the results are shown as open circles in Fig. 1. We see that the field-theoretic and collective-coordinate approaches for $\omega^{(1)}(k)$ are consistent.

For the dilute Bose gas we must include the $O(g)$ term in (4.14) and write

$$\omega^{(1)}(k) = \omega_1(k) + \omega_2(k) + k^2 v^{(1)}/2\omega^{(0)}(k) \, . \quad (4.34)$$

Then in the same manner as above it is easy to take the small-k limit of ω_1 and ω_2 for $\omega^{(0)}(k)$ given by (3.33) and to recover (4.24) of Ref. 4.

C. Static structure function

We determine the static structure function by employing a method analogous to that used by Bhattacharyya and Woo.[5] The method is to obtain an expression for $\omega(k)$ that is related to $S(k)$, set this expression equal to our previous result for $\omega^{(1)}(k)$ and hence determine $S(k)$ to $O(g)$.

Let us define a new set of multiexcitation wave functions by

$$|\vec{k}) = [S(k)]^{-1/2}\rho(k)|0)$$
$$|\vec{k}_1, \vec{k}_2) = [S(k_1)S(k_2)]^{-1/2}\rho(k_1)\rho(k_2)|0) \, , \quad (4.35)$$

etc., where $|0)$ is the exact ground-state wave function and $S(k) = (0|\rho^\dagger(k)\rho(k)|0)$ [the latter definition of $S(k)$ is equivalent to (3.44) except for the different normalization]. We consider $\delta H = H - (\vec{k}|H|\vec{k})$ as the perturbation and find, using standard perturbation theory, that

$$\omega(k) = \omega_F(k) + \epsilon_2(k) \, , \quad (4.36)$$

where

$$\omega_F(k) = (0|\delta H|0) = k^2/2S(k) \quad (4.37)$$

and

$$\epsilon_2(k) = \frac{1}{2} \sum_{k, p \neq 0} \frac{|(\vec{k}-\vec{p}, \vec{p}|\delta H|\vec{k})|^2}{\omega_F(k) - \omega_F(p) - \omega_F(|\vec{k}-\vec{p}|)} \, . \quad (4.38)$$

To evaluate $\epsilon_2(k)$ to $O(g)$ we need retain terms in δH to $O(g^{1/2})$. We write

$$(\vec{k}-\vec{p}, \vec{p}|\delta H|\vec{k}) = (\vec{k}-\vec{p}, \vec{p}|H_0+H_1-E_0|\vec{k})$$
$$- \omega_0(k)(\vec{k}-\vec{p}, \vec{p}|\vec{k}) \, , \quad (4.39)$$

where the second matrix element in (4.39) is frequently referred to as the overlap integral. We expand to first order in H_1 and write

$$|0) = |0\rangle + \frac{H_1}{H_0 - E_0}|0\rangle + \cdots \, , \quad (4.40)$$

where $|0\rangle$ is the vacuum state defined in Sec. IV B. We use (4.15), (4.17), (4.18), and (4.40) to write the matrix elements in (4.39) as

$$(\vec{k}-\vec{p}, \vec{p}|H_0+H_1-E_0|\vec{k}) = \frac{1}{2}g^{1/2}[\lambda_k\lambda_p\lambda_{|\vec{k}-\vec{p}|}]^{-1/2}[\vec{k}\cdot\vec{p}\lambda_{|\vec{k}-\vec{p}|} + \vec{k}\cdot(\vec{k}-\vec{p})\lambda_p] \, , \quad (4.41)$$

$$(\vec{k}-\vec{p}, \vec{p}|\vec{k}) = \frac{1}{2}g^{1/2}(\lambda_k\lambda_p\lambda_{|\vec{k}-\vec{p}|})^{-1/2}[k^2/2\lambda_k + (|\vec{k}-\vec{p}|)^2/2\lambda_{|\vec{k}-\vec{p}|} + p^2/2\lambda_p]^{-1}$$

$$\times[-(\vec{k}\cdot\vec{p})\lambda_{|\vec{k}-\vec{p}|}(1+\lambda_k\lambda_p) - \vec{k}\cdot(\vec{k}-\vec{p})\lambda_p(1+\lambda_p\lambda_{|\vec{k}-\vec{p}|}) - \vec{p}\cdot(\vec{k}-\vec{p})\lambda_k(1+\lambda_p\lambda_{|\vec{k}-\vec{p}|})] \, , \quad (4.42)$$

and obtain

$$(\vec{k}-\vec{p}, \vec{p}|\delta H|\vec{k}) = \frac{1}{2}g^{1/2}(\lambda_k\lambda_p\lambda_{|\vec{k}-\vec{p}|})^{-1/2}\lambda_k[p^2\lambda_k\lambda_{|\vec{k}-\vec{p}|} + (\vec{k}-\vec{p})^2\lambda_k\lambda_p + k^2\lambda_p\lambda_{|\vec{k}-\vec{p}|}]^{-1}\{[k^2+p^2+(\vec{k}-\vec{p})^2]$$

$$\times[\vec{k}\cdot(\vec{k}-\vec{p})\lambda_p^2(1-\lambda_{|\vec{k}-\vec{p}|}^2) + (\vec{k}\cdot\vec{p})\lambda_{|\vec{k}-\vec{p}|}^2(1-\lambda_p^2)] - 2(\vec{k}\cdot\vec{p})\vec{k}\cdot(\vec{k}-\vec{p})(\lambda_p - \lambda_{|\vec{k}-\vec{p}|})^2\} \, , \quad (4.43)$$

where λ_k is defined in (3.22). We obtain $\epsilon_2(k)$ to $O(g)$ by substituting (4.43) into (4.38) and replacing $\omega_F(k)$ by $\omega^{(0)}(k)$.

Because of the factor $N^{-1/2}$ introduced in the definition (4.1) of $\rho(k)$ we write $S(k)$ to $O(g)$ as

$$S(k) = S^{(0)}(k) + gS^{(1)}(k) \quad (4.44)$$

so that $S^{(0)}$ and $S^{(1)}$ have the same meaning as in (3.45). We substitute (4.44) into (4.37) and use (4.36) to write $\omega(k)$ to $O(g)$

$$\omega(k) = \omega^{(0)}(k) + g\epsilon_2(k) + g[-k^2 S^{(1)}(k)/2S^{(0)}(k)^2] \, . \quad (4.45)$$

In (4.45) we have explicitly shown the $O(g)$ depen-

dence of ϵ_2. We now compare the first-order terms in (4.45) with the previous result (4.21) for $\omega^{(1)}(k)$ and obtain the result for $S^{(1)}(k)$

$$S^{(1)}(k) = -2S^{(0)}(k)^2/k^2[\omega_1(k) + \omega_2(k) - \epsilon_2(k)] \, , \quad (4.46)$$

where the integrals ω_1, ω_2, and ϵ_2 all involve $S^{(0)}(k)$. We have thus shown that the collective coordinate method allows us to calculate both $\omega^{(1)}(k)$ and $S^{(1)}(k)$ in a simple manner.

We limit ourselves to showing that for the charged Bose gas $S^{(1)}(k)$ in (4.46) agrees analytically for small k with the field-theoretic result (3.47). We take the $k \to 0$ limit of (4.43) and (3.27)

and find that

$$\epsilon_2(k) = \frac{k^4 23}{120\pi^2} \int_0^\infty dp \, \frac{p^2[1-(1+\frac{1}{4}p^2)^{1/2}]}{(1+\frac{1}{4}p^4)(p^4+3)^2} \,. \quad (4.47)$$

If we combine (4.47) with (4.29) and (4.32) and use (4.46), we recover the result (3.47). For arbitrary k we numerically evaluate (4.46) for the charged Bose gas. The results are shown as open circles in Fig. 1. We see that the two approaches are consistent.

We noted in Sec. IV B that because the term H_2 was omitted in (4.6) SYK^2 could not obtain a divergence-free $\omega(k)$ using standard perturbation theory. They consequently calculated $\omega(k)$ in a manner similar to that used in this section and obtained a result in the form (4.45). However they did not include the $O(g)$ term involving $S^{(1)}(k)$, and hence their calculation of $\omega(k)$ is not correct to $O(g)$.

V. CORRELATED-BASIS-FUNCTION APPROACH

The basic assumption of the correlated-basis-function[1] (CBF) approach is that the wave functions

$$\Psi = \psi \prod_i \rho(k_i) \qquad (5.1)$$

are useful in describing the ground state and low-lying excited states. The correlation factor ψ in-

cludes the dynamical correlations between the particles and is chosen to be a symmetrical function of the coordinates in order to insure the over-all symmetry of Ψ. The two most widely used forms for ψ are the Bijl-Dingle-Jastrow (BDJ) form and the exact ground-state wave function.

A. Excitation spectrum

The BDJ choice of ψ can be used together with the variational principle and was used by Lee and Feenberg[12] to calculate the "optimum" but approximate excitation energy in the form $\omega(k) = \omega^{(0)}(k) + g\omega_v(k)$ where

$$\omega_v(k) = -\frac{k^2 S(k)}{8} \int \frac{d^3p}{(2\pi)^3}\left(1+\frac{2p^2}{k^2}\right)$$
$$\times [1-S(p)][1-S(|\vec{k}+\vec{p}|)]\,, \quad (5.2)$$

with $\omega^{(0)}(k)$ given in (3.13). The optimum structure function is defined as $S(k) = \langle 0|\rho^\dagger(k)\rho(k)|0\rangle$, where $|0\rangle$ represents the BDJ wave function. The leading $O(g)$ correction to (5.2) arises from three excitation processes and was calculated by Bhattacharyya and Woo (BW).[5] We combine their result with (5.2) and write

$$\omega^{(1)}(k) = \omega_v(k) + \omega_p(k)\,, \qquad (5.3)$$

where

$$\omega_p(k) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{|\langle \vec{k}-\vec{p},\vec{p}|\delta H|\vec{k}\rangle|^2}{\omega^{(0)}(k)-\omega^{(0)}(p)-\omega^{(0)}(|\vec{k}-\vec{p}|)} - \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{|\langle \vec{k},\vec{p},-\vec{k}-\vec{p}|\delta H|0\rangle|^2}{\omega^{(0)}(k)+\omega^{(0)}(p)+\omega^{(0)}(|\vec{k}+\vec{p}|)}\,, \qquad (5.4)$$

with

$$\langle \vec{k}-\vec{p},\vec{p}|\delta H|\vec{k}\rangle = \frac{1}{4}[S(k)S(p)S(|\vec{k}-\vec{p}|)]^{-1/2}[\vec{k}\cdot(\vec{k}-\vec{p})S(p)+\vec{p}\cdot(\vec{k}-\vec{p})S(k)+\vec{k}\cdot\vec{p}S(|\vec{k}-\vec{p}|)-k^2 S(p)S(|\vec{k}-\vec{p}|)$$
$$+(\vec{k}-\vec{p})^2 S(k)S(p)+p^2 S(k)S(|\vec{k}-\vec{p}|)-(k^2+p^2-\vec{k}\cdot\vec{p})S(k)S(p)S(|\vec{k}-\vec{p}|)]\,, \qquad (5.5)$$

$$\langle \vec{k},\vec{p},-\vec{k}-\vec{p}|\delta H|0\rangle = \frac{1}{4}[S(k)S(p)S(|-\vec{k}-\vec{p}|)]^{-1/2}[\vec{k}\cdot(-\vec{k}-\vec{p})S(p)+\vec{k}\cdot\vec{p}S(|\vec{k}+\vec{p}|)+(\vec{k}+\vec{p})\cdot\vec{p}S(k)$$
$$+k^2 S(|\vec{k}+\vec{p}|)S(p)+(\vec{k}+\vec{p})^2 S(k)S(p)+p^2 S(|\vec{k}+\vec{p}|)S(k)-(k^2+p^2+\vec{k}\cdot\vec{p})S(k)S(p)S(|\vec{k}+\vec{p}|)]\,. \qquad (5.6)$$

In the above $|k,\,k'\rangle$ and $|k,\,k',\,k''\rangle$ are defined as in (4.35), $S(k)$ is the optimum structure function, and $\delta H = H - \langle k|H|k\rangle$. For the dilute Bose gas there is the additional contribution $k^2 v^{(1)}/2\omega^{(0)}(k)$ in (5.3) [see (4.34)].

Since we are interested in $\omega_v(k)$ and $\omega_p(k)$ to $O(g)$, we can replace $S(k)$ in (5.3), (5.5), and (5.6) by λ_k, (3.46). BW have shown that the resulting integral for $\omega^{(1)}(k)$ agrees numerically for all k up to $k = 2$ with the field-theoretic result shown in Fig. 1. We show here that the $O(g)$ result (5.3) agrees analytically for all k with $\omega^{(1)}(k)$ (4.21) found by collective coordinate method. We write the matrix elements (4.27) and (4.28) in terms of (5.5) and (5.6):

$$\langle \vec{k}-\vec{p},\vec{p}|H_1|\vec{k}\rangle = \langle \vec{k}-\vec{p},\vec{p}|\delta H|\vec{k}\rangle + \frac{1}{2}(\lambda_k\lambda_p\lambda_{|\vec{k}-\vec{p}|})^{1/2}[\omega^{(0)}(k)-\omega^{(0)}(p)-\omega^{(0)}(|\vec{k}-\vec{p}|)]\,, \qquad (5.7)$$

$$\langle \vec{k},\vec{p},-\vec{k}-\vec{p}|H_1|0\rangle = \langle \vec{k},\vec{p},-\vec{k}-\vec{p}|\delta H|0\rangle - \frac{1}{2}(\lambda_k\lambda_p\lambda_{|\vec{k}+\vec{p}|})^{1/2}[\omega^{(0)}(k)-\omega^{(0)}(p)-\omega^{(0)}(|\vec{k}+\vec{p}|)]\,. \qquad (5.8)$$

If we substitute (5.7) and (5.8) into (4.25) and compare with (5.2) and (5.4), we find

$$\omega_2(k) = \omega_p(k) + \omega_v(k) + \Delta(k) , \qquad (5.9)$$

where

$$\Delta(k) = - \frac{\lambda_k \lambda_p \lambda_{|\vec{k}-\vec{p}|}}{4[\omega^{(0)}(p) + \omega^{(0)}(|\vec{k}-\vec{p}|)]} . \qquad (5.10)$$

A simple rearrangement of terms in (5.10) gives $\Delta(k) = -\omega_1(k)$ and we arrive at the desired result that

$$\omega_1(k) + \omega_2(k) = \omega_v(k) + \omega_p(k) . \qquad (5.11)$$

We thus conclude that the $O(g)$ results for the excitation spectrum using the field-theoretic, collective-coordinate, and correlated-basis-function approaches are consistent.

B. Static structure function

We discuss here Bhattacharyya and Woo's calculation of the static structure function. We follow the approach of Sec. IV C and take the correlation factor ψ to be the exact ground-state wave function $|0\rangle$. The excitation spectrum is given in the form of a correction to the Feynman excitation spectrum:

$$\omega(k) = \omega_F(k) + \hat{\epsilon}_2(k) , \qquad (5.12)$$

where

$$\hat{\epsilon}_2(k) = \frac{1}{2} \sum_{p \neq 0, k} \frac{|(\vec{k}-\vec{p}, \vec{p}|\delta H|\vec{k})|^2}{\omega_F(k) - \omega_F(p) - \omega_F(|\vec{k}-\vec{p}|)} , \qquad (5.13)$$

$$(\vec{k}-\vec{p}, \vec{p}|\delta H|\vec{k}) = (\vec{k}-\vec{p}, \vec{p}|H - E|\vec{k})$$
$$- \omega_F(k)(\vec{k}-\vec{p}, \vec{p}|\vec{k}) . \qquad (5.14)$$

In the above, $\delta H = H - (\vec{k}|H|\vec{k})$, $H|0\rangle = E|0\rangle$, and $\omega_F(k)$ is defined in (4.37). The matrix element $(\vec{k}-\vec{p}, \vec{p}|H - E|\vec{k})$ can be evaluated exactly and has the form

$$(\vec{k}-\vec{p}, \vec{p}|H - E|\vec{k}) = \tfrac{1}{2} g^{1/2} [S(k)S(p)S(|\vec{k}-\vec{p}|)]$$
$$\times [\vec{k} \cdot (\vec{k}-\vec{p})S(p) + \vec{k} \cdot \vec{p} S(|\vec{k}-\vec{p}|)] . \qquad (5.15)$$

The overlap integral $(\vec{k}-\vec{p}, \vec{p}|\vec{k})$ in (5.14) involves the three-particle distribution function $p^{(3)}(1, 2, 3)$,

$$(\vec{k}-\vec{p}, \vec{p}|\vec{k}) = \tfrac{1}{2} g^{1/2} [S(k)S(p)S(|\vec{k}-\vec{p}|)]^{-1/2}$$
$$\times [-2 + S(k) + S(p) + S(|\vec{k}-\vec{p}|)]$$
$$+ g \int d\vec{r}_{123} e^{i\vec{k}\cdot\vec{r}_{12} + i\vec{p}\cdot\vec{r}_{23}} p^{(3)}(1, 2, 3) . \qquad (5.16)$$

Since the form of $p^3(1, 2, 3)$ is not known, BW followed the approach of Jackson and Feenberg[12] and made the convolution approximation[1,7] for $p^3(1, 2, 3)$ in (5.16). With this assumption they found that (5.16) can be written as

$$(\vec{k}-\vec{p}, \vec{p}|\vec{k}) = g^{1/2} [S(k)S(p)S(|\vec{k}-\vec{p}|)]^{1/2} . \qquad (5.17)$$

In (5.15)–(5.17) the $S(k)$ is the exact static structure function.

Since we need to obtain $\omega(k)$ to $O(g)$, we can replace $\omega_F(k)$ in (5.13) by $\omega^{(0)}(k)$ (3.13) and $S(k)$ in (5.15) and (5.17) by λ_k (3.46). We use (4.45) to write $\omega_F(k)$ to $O(g)$ and write (5.12) to $O(g)$ as

$$\omega(k) = k^2/2S^{(0)}(k) + g\hat{\epsilon}_2(k) + g[-k^2 S^{(1)}(k)/2S^{(0)}(k)^2] . \qquad (5.18)$$

We have included the factor of g with $\hat{\epsilon}_2$. We now compare the $O(g)$ terms in (5.18) with the $O(g)$ terms in (5.3) and determine $S^{(1)}(k)$. We find

$$S^{(1)}(k) = - \frac{2S^{(0)}(k)^2}{k^2 [\omega_v(k) + \omega_p(k) - \hat{\epsilon}_2(k)]} . \qquad (5.19)$$

BW evaluated (5.19) numerically for the charged Bose gas. Their result is shown as the dotted line in Fig. 2 and is seen to be inconsistent with the results (3.47) and (4.46) for $S^{(1)}(k)$ obtained by both field-theoretic and collective-coordinate approaches. To find the source of the error in (5.19) we note that the error does not arise from our form for $\omega_v(k) + \omega_p(k)$, since it was shown in Sec. V B that $\omega_v(k) + \omega_p(k)$ gives the correct result for $\omega^{(1)}(k)$. If we use (4.46), (5.11), and (5.19) we see that the error in (5.19) must arise from $\hat{\epsilon}_2(k)$. A comparison of (4.38) and (5.13) and (4.41) and (5.15) shows that $\epsilon_2(k)$ and $\hat{\epsilon}_2(k)$ have the same form and that the matrix elements (4.41) and (5.15) are identical to first order in g. However, we see that the overlap matrix elements (4.42) and (5.17) are not equivalent. Therefore we conclude that the convolution approximation, which leads us from the exact form (5.16) to the approximate form (5.17), leads to incorrect results to $O(g)$ and that the correct form to $O(g)$ of the overlap matrix element is given in (4.42).

VI. DISCUSSION

We first summarize the results of the three independent approaches of Secs. III–V. The results for the first-order excitation spectrum $\omega^{(1)}(k)$ obtained in the field-theoretic, collective-coordinate, and correlated-basis-function (with the choice of the Bijl-Dingle-Jastrow form of the correlation factor) formalisms were shown to be consistent with one another. The k dependence of $\omega^{(1)}(k)$ for the charged Bose gas is shown in Fig. 1. It was shown that in the limit of small k, $\omega^{(1)}(k)$ is an analytic function of k through at least order k^6 for the charged Bose gas in contrast to the nonanalytic behavior of $\omega^{(1)}(k)$ for the dilute Bose gas. The analyticity of $\omega^{(1)}(k)$ for the charged Bose gas can be associated with the fact that the denominator $\omega^{(0)}(k) - \omega^{(0)}(p) - \omega^{(0)}(|\vec{p}-\vec{k}|)$, which appears in various integrals for $\omega^{(1)}(k)$ [see for example (3.21)], does not vanish in the limit of small k for

the charged Bose gas as it does for the dilute Bose gas.

The dielectric-function approach allowed us to isolate the backflow and continuum contributions to the first-order static structure function $S^{(1)}(k)$. It was found that for the charged Bose gas the leading k^4 dependence of both the backflow and continuum contributions to $S^{(1)}(k)$ lead to the breakdown of the single excitation Feynman relation $S(k) = k^2/2m\omega(k)$ at order k^4 in contrast to the dilute Bose gas[4] for which leading k dependence of the backflow and continuum contributions is order k^3 and k^4, respectively. Recently Wong[13] has presented simple sum-rule arguments for the leading k dependence of the backflow contribution to the static structure function of neutral and charged quantum liquids. Our microscopic calculation is consistent with his general conclusions. The k dependence of $S^{(1)}(k)$ for the charged Bose gas obtained by both the dielectric-function and collective-coordinate approaches is shown in Fig. 2. The dielectric-function approach also allows us to determine the form of the first-order dynamic structure function $S^{(1)}(k, \omega)$. It was found that for the charged Bose gas $S^{(1)} \propto k^4 \omega^{-11/2}$ in the limit of high frequencies. We conjecture that this limiting form is independent of perturbation theory.

The present study of simple models of Bose systems using the field-theoretic, collective-coordinate, and correlated-basis-function formalisms has given us some insight into the assumptions that have been used by various workers in applying the latter two formalisms to superfluid ^{4}He. A comparison of the work of Sunakawa, Yamasaki, and Kebukawa[2] with the model collective-coordinate calculation of Sec. IV shows that they neglected a term that is the same order of magnitude as the terms they considered. Thus, the fact that their calculated excitation spectrum is consistent with experiment appears to be dependent on their use of an effective potential with three parameters.

We were able to exploit the flexibility of the collective-coordinate formalism and calculate $S^{(1)}(k)$ in an analogous manner to the correlated-basis-function approach of Bhattacharyya and Woo.[5] It was shown that their result for $S^{(1)}(k)$ is incomplete to first order in g and that the source of the error

is the use of the convolution approximation[1] for the three-particle distribution function in the evaluation of the overlap integral $(\vec{k} - \vec{p}, \vec{p} | \vec{k})$ [see (5.16)]. A comparison of the two calculations leads us to propose the form (4.42) for the overlap integral. The incorrect form (5.17) for the overlap integral was used by Jackson and Feenberg[3] in their calculation of the excitation spectrum of superfluid ^{4}He. It would be desirable to perform their calculation using the form (4.42) for the overlap integral with λ_k replaced by experimental $S(k)$.

Berdahl[14] has obtained the correct form (4.42) for the overlap integral by a functional differentiation of the ground-state energy. The advantage of the present approach is that it follows closely the original calculations of Jackson and Feenberg[3] and Bhattacharyya and Woo[5] and allows us to obtain the correct form of the overlap integral directly. Grest and Rajopal[10] have used the Bogoliubov-Zubarev[9] method to determine $S^{(1)}(k)$.

The application of the collective-coordinate and correlated-basis-function approaches to superfluid ^{4}He have been limited to zero temperature. Since the temperature dependence of $\omega(k)$ and $S(k)$ is of interest, it would be desirable to extend the present model calculations of $\omega^{(1)}(k)$ and $S^{(1)}(k)$ to nonzero temperature so that we will have a guide to more realistic calculations. Also, since the first-order investigations have uncovered unexpected physical features, it would be of interest to consider a model calculation in which four-phonon processes are included.

Note added in proof. One of the authors (F. F.) has recently used a different field-theoretic approach to consider the high-frequency behavior of $S(k, \omega)$ for quantum fluids,[15] and finds his results for the Bose system to agree with that of Sec. III C. He has also shown that the divergence of the fifth frequency moment for a charged Bose gas, which is implied by (3.41), to be independent of perturbation theory.

ACKNOWLEDGMENTS

*Partially supported by a grant from the Research Corporation.

†Work based on a part of a Ph. D. thesis submitted by F. Family to Clark University (June, 1974).

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