

Kinetic-theory approach to normal Fermi liquids

Oriol T. Valls and Gene F. Mazenko*

James Franck Institute and Department of Physics, The University of Chicago, Chicago, Illinois 60637

Harvey Gould

Department of Physics, Clark University, Worcester, Massachusetts 01610

(Received 14 March 1977)

We generalize the fully renormalized kinetic-theory formalism of Mazenko to normal quantum fluids and show that the Kubo linear-response function satisfies a generalized kinetic equation with a memory function or kernel nonlocal in space and time. In contrast to classical systems, the generalized kinetic equations must be supplemented by a "boundary-condition" equation for the static part of the response function. The memory function is separated in a natural way into a static and dynamic part; only the static part appears in the boundary-condition equation. We consider a normal Fermi liquid at low temperatures and obtain the correspondence, in the limit of small wave vector k and small frequency ω , between the formal expressions for the memory function and the phenomenological Landau theory. From this analysis and the use of kinetic modeling we develop a simple model for the memory function which is applicable to higher k and ω , includes collisional effects, and is consistent with the conservation laws. The dynamic-structure function $S(k, \omega)$ is obtained from the generalized kinetic equation and evaluated for liquid ^3He at low temperatures using the effective mass, the static-structure function, and the viscosity as input. The model results for $S(k, \omega)$ and the dispersion relation of zero sound contain no free parameters and are consistent with recent inelastic-neutron-scattering measurements on liquid ^3He .

I. INTRODUCTION

Recent experiments in normal liquid ^3He and the continued strong interest in superfluid ^3He give new impetus to the development of a microscopic theory of the dynamical properties of a normal Fermi liquid. In this paper, we focus our attention on the inelastic neutron scattering experiments on liquid ^3He performed at Grenoble¹ and Argonne.² These experiments determine the dynamic-structure³ function $S(k, \omega)$ at low temperatures for values of the wave vector k and frequency ω such that it is unreasonable to expect the phenomenological Landau Fermi liquid theory⁴ to be applicable. The transverse zero sound measurements⁵ in normal ^3He are also of interest, since they appear to be unexplained by the Landau theory.

Another motivation for an investigation of the dynamics of realistic quantum fluids is the success of recent developments in classical kinetic theory. For example, the fully renormalized kinetic-theory formalism of Mazenko⁶ has been successful in explaining the qualitative features of the dynamics of classical fluids over a wide range of densities, and has incorporated nonlocal effects in space and time which can be probed by large momentum and energy transfer neutron scattering experiments. In the past, the success of such an investigation of quantum fluids appeared doubtful when the dynamics of classical fluids was not well understood.

The main goal of this paper is to develop new

methods for calculating equilibrium averaged time-dependent correlation functions such as $S(k, \omega)$ for normal Fermi liquids at low temperatures. The usual approach to the calculation in quantum systems consists essentially of establishing a perturbation scheme, usually with the aid of diagrams, to solve approximately the hierarchy of equations for the Green's functions.⁷ In addition to the technical difficulties associated with the complexity of any nontrivial approximation, this approach presents the following problems: It is difficult to insure⁸ that a given approximation is consistent with the conservation laws, sum rules, and general symmetry considerations; the nature of the Green's-function hierarchy is such that it is difficult to make approximations on the basis of physical insight, study of known limits, or relations to phenomenological theories; several time variables must be introduced in the intermediate stages of the calculation and then eliminated to obtain the desired result which depends on one time variable only. The introduction of several time variables is the most fundamental problem associated with the Green's-function method since it implies that the intrinsically static effects cannot be separated from the dynamical effects in a natural way.

These difficulties have restricted the success of attempts to calculate $S(k, \omega)$ over a wide range of k , ω , and temperature for realistic quantum systems. One of us (G.M.),⁹ has been successful in using a Green's-function method to establish a low-

density kinetic equation from which $S(k, \omega)$ could be obtained for all k and ω . The analysis was very cumbersome even in the high-temperature classical limit, but it did lead to further work on the dynamics of classical systems. Kadanoff and Baym¹⁰ have given a Green's-function method for obtaining kinetic equations for systems slowly varying in space and time, but there remains a need for a quantum kinetic theory valid on all scales of length and time.

Our approach to quantum kinetic theory is based on the extension to quantum systems of the fully renormalized kinetic-theory (FRKT) formalism.⁶ The FRKT formalism has been discussed by Mazenko¹¹ for quantum and classical systems, and first applied to quantum systems by Boley and Smith.¹² Although much of the formal development to be presented in this paper is similar to that of Boley and Smith, the method of application of the FRKT will be more similar in spirit to the original formulation of Mazenko. The advantages of the FRKT formalism for both classical and quantum systems include: the kinetic equation depends on one time only; renormalization effects are included in a natural way by the introduction of a memory function or kernel in the kinetic equation and by the introduction of an effective two-body interaction; the conservation laws, sum rules, and symmetries can be easily related to various properties of the memory function; the physical significance of the different terms in the kinetic equation is usually sufficiently clear so that approximations can be made with the aid of physical insight; dynamic effects are separated from intrinsically static effects so that the static correlation functions can be included in a systematic way. However, the development of a successful quantum kinetic theory is not easy even with these simplifications. As will be discussed in the following, approximation procedures for quantum systems are considerably more difficult than for classical systems.

In Sec. II, we discuss the general formalism and introduce the correlation functions of interest. Although it is possible to generalize the FRKT formalism to include Bose condensation and pairing, we do not discuss these cases here. We show in Sec. III that the Kubo linear response function satisfies a generalized kinetic equation with a memory function nonlocal in space and time. It is shown that in contrast to classical kinetic theory, the generalized kinetic equation must be supplemented by a "boundary-condition" equation for the static part of the response function. The memory function is separated in a natural way into a static and dynamic part; only the static part of the memory function appears in the boundary-condition equation.

In Sec. IV, we consider a normal Fermi liquid at low temperatures. The correspondence between the formal expressions for the memory function and the phenomenological Landau theory is obtained in the limit of small k and ω . This correspondence yields a relation between the Landau parameters and the microscopic theory which we believe to be simpler than the usual Bethe-Salpeter equation approach.¹³ From this analysis of the Landau limit and a discussion of kinetic modeling in the classical limit, we develop a simple model for the memory function which is applicable to higher k and ω , includes collisional effects, and is consistent with the conservation laws. The dynamic-structure function $S(k, \omega)$ is obtained from the generalized kinetic equation and is evaluated for liquid ³He at low temperature. We conclude in Sec. V with a discussion of our results and their relation to the neutron scattering experiments.

II. FORMALISM

A. Phase-space operators

The system of interest is a one-component monoatomic quantum fluid of mass m particles at temperature $T = (k_B \beta)^{-1}$ and density n . For simplicity of notation, we assume a unit volume and neglect particle spin unless otherwise noted. Classical kinetic theory is developed in terms of the phase space densities. For example, the single-particle phase space density is defined classically by

$$f_c(rp, t) = \sum_i \delta(r - R_i(t)) \delta(p - P_i(t)), \quad (2.1)$$

where R_i, P_i are the phase space coordinates of particle i , and the summation includes all particles. The natural generalization of (2.1) to the quantum case, for which the position and momentum of a particle cannot be defined simultaneously, is given by the Wigner operator¹⁴

$$f(rp, t) = (2\pi\hbar)^{-3} \int d^3r' e^{-i\vec{p} \cdot \vec{r}'/\hbar} \times \psi^\dagger(r - \tfrac{1}{2}\vec{r}', t) \psi(r + \tfrac{1}{2}\vec{r}', t). \quad (2.2)$$

In the above,¹⁵ $\psi(r, t)$ and $\psi^\dagger(r, t)$ are the Heisenberg field operators, and satisfy the usual commutation relations for fermions or bosons.

Multiparticle density operators are defined by¹²

$$f(12 \dots n, t) = \int d^3r'_n e^{-i\vec{p}_n \cdot \vec{r}'_n/\hbar} \psi^\dagger(r_n - \tfrac{1}{2}\vec{r}'_n, t) \times f(12 \dots n-1, t) \psi(r_n + \tfrac{1}{2}\vec{r}'_n, t), \quad (2.3)$$

where $1 \equiv (r_1 p_1)$, etc. The density operators are Hermitian and symmetric under interchange of any pair of particles for both fermions and bosons. All of the microscopic quantities of interest can be ex-

pressed in terms of the one-particle and two-particle density operators. For example, the Hamiltonian of the system is given by

$$H = \int d^3r \frac{p^2}{2m} f(1) + \frac{1}{2} \int d^3r_1 d^3r_2 v(12) f(12), \quad (2.4)$$

where $v(12) = v(|\vec{r}_1 - \vec{r}_2|)$ is the interparticle potential.

The equal-time commutation relations of the first few Wigner operators can be expressed as¹²

$$[f(1), f(2)] = \delta(1-2)2i \\ \times \sin[\frac{1}{2}\hbar(\vec{\nabla}_{r_1} \cdot \vec{\nabla}_{p_1} - \vec{\nabla}_{p_1} \cdot \vec{\nabla}_{r_1})] f_1(1), \quad (2.5a)$$

$$[f(1), f(23)] = \delta(1-2)2i \\ \times \sin[\frac{1}{2}\hbar(\vec{\nabla}_{r_1} \cdot \vec{\nabla}_{p_1} - \vec{\nabla}_{p_1} \cdot \vec{\nabla}_{r_1})] f(13) \\ + \delta(1-3)2i \\ \times \sin[\frac{1}{2}\hbar(\vec{\nabla}_{r_1} \cdot \vec{\nabla}_{p_1} - \vec{\nabla}_{p_1} \cdot \vec{\nabla}_{r_1})] f(12). \quad (2.5b)$$

The arrows indicate whether the gradient operators act to their left or to their right. These operators are awkward to use; however note that after a Fourier transform with respect to r , an operator such as $e^{-i\hbar \vec{\nabla}_r \cdot \vec{\nabla}_p}$ becomes a momentum displacement operator if it acts on a momentum-dependent function to its right. We have for any function $g(p)$

$$e^{\hbar \vec{k} \cdot \vec{\nabla}_p} g(p) = g(p + \hbar k).$$

It is thus convenient to introduce the spatial Fourier transformed operators

$$f_k(p) = \int d^3r e^{-i\vec{k} \cdot \vec{r}} f(rp), \quad (2.6a)$$

$$f_{k_1 k_2}(p_1 p_2) = \int d^3r_1 d^3r_2 e^{-i\vec{k}_1 \cdot \vec{r}_1} e^{-i\vec{k}_2 \cdot \vec{r}_2} f(12). \quad (2.6b)$$

We can rewrite the equal-time commutation relations (2.5) in terms of f_k and $f_{k_1 k_2}$. If we take into account translational invariance, the commutation relations of interest include

$$[f_k(p), f_{-k}(p')] = \delta(p-p')[f_0(p-\hbar k/2) - f_0(p+\hbar k/2)], \quad (2.7a)$$

$$[f_k(p), f_{k_1, -k-k_1}(p_1, p_2)] = \delta\left(p-p_2-\frac{\hbar k_1}{2}\right) f_{-k_1, k_1}\left(\frac{p+p_2}{2}-\hbar\frac{(2k+k_1)}{4}, p_1\right) \\ - \delta\left(p-p_2+\frac{\hbar k_1}{2}\right) f_{-k_1, k_1}\left(\frac{p+p_2}{2}+\hbar\frac{(2k+k_1)}{4}, p_1\right) \\ + \delta\left(p-p_1+\hbar\frac{k+k_1}{2}\right) f_{k+k_1, -k-k_1}\left(\frac{p+p_1}{2}-\hbar\frac{k-k_1}{4}, p_2\right) \\ - \delta\left(p-p_1-\hbar\frac{k+k_1}{2}\right) f_{k+k_1, -k-k_1}\left(\frac{p+p_1}{2}+\hbar\frac{k-k_1}{4}, p_2\right). \quad (2.7b)$$

Note that the commutation relations (2.5) and (2.7) are independent of the statistics.

We can write the usual Heisenberg equation of motion for the Wigner operator in the form

$$\frac{\partial f(1, t)}{\partial t} = iL f(1, t), \quad (2.8)$$

where L is the Liouville operator

$$L = (1/\hbar)[H, \quad]. \quad (2.9)$$

The time dependence of $f(1, t)$ can be found by a formal integration of (2.8)

$$f(1, t) = e^{iLt} f(1). \quad (2.10)$$

The effect of the Liouville operator follows directly from the commutation relations. It is easy to show that

$$L f(1, t) = -L_0(1) f(1, t) - \int d\bar{1} L_I(1\bar{1}) f(1\bar{1}, t), \quad (2.11a)$$

where $L_0(1)$ is the single-particle Liouville operator

$$L_0(1) = -i\vec{p}_1 \cdot \vec{\nabla}_{r_1}/m, \quad (2.12a)$$

and $L_I(12)$ is the interaction part of the two-particle Liouville operator

$$L_I(1\bar{1}) = i v(1\bar{1}) \frac{2}{\hbar} \sin(\frac{1}{2}\hbar \vec{\nabla}_{r_1} \cdot \vec{\nabla}_{p_1}). \quad (2.13a)$$

If we Fourier transform the spatial coordinates, we obtain

$$\frac{i\partial f_k(p, t)}{\partial t} = -L_0(kp) f_k(p, t) \\ - L_I(k\bar{k}, p) f_{\bar{k}, -\bar{k}}(p\bar{p}, t), \quad (2.11b)$$

where

$$L_0(kp) = (k \cdot p)/m, \quad (2.12b)$$

and

$$L_I(kk', p) = -(2/\hbar)v(k') \sinh(\frac{1}{2}\hbar k \cdot \vec{\nabla}_p). \quad (2.13b)$$

Barred indices such as \bar{k} in (2.11b) imply an integration¹⁶ over that variable; $v(k)$ in (2.13b) is the spatial Fourier transform of $v(r)$.

B. Correlation functions

The physical quantities of interest can be expressed as equilibrium averages of the density operators. For example, the one-particle distribution function is given by

$$n(1) = \langle f(1) \rangle, \quad (2.14)$$

where $\langle \dots \rangle$ denotes the thermal average in the grand canonical ensemble specified by β and the chemical potential μ . Because of translational invariance, $n(1)$ becomes simply the momentum distribution. The normalization condition on $n(p)$ is $(2\pi\hbar)^{-3} \int d^3p n(p) = n$. For free fermions, $n(p)$ reduces to

$$n^0(p) = [e^{\beta(p^2/2m - \mu)} + 1]^{-1}. \quad (2.15)$$

The pair distribution function $g(r)$ is defined by

$$n^2 g(r_1 - r_2) = \int d^3p_1 d^3p_2 n(12), \quad (2.16)$$

where $n(12) = \langle f(12) \rangle$. The static-structure function $S(k)$ can be determined by x-ray scattering measurements,¹⁷ and related to $g(r)$ by

$$S(k) - 1 = n \int d^3r e^{i\vec{k} \cdot \vec{r}} [g(r) - 1]. \quad (2.17)$$

The linear response and transport properties of a fluid can be described by time-dependent equilibrium averaged correlation functions. For a classical system, it is natural to study the phase-space density correlation function

$$S_c(11', t - t') = \langle \delta f(1, t) \delta f(1', t') \rangle.$$

In the quantum case, there are several interesting two-point correlation functions of the one-particle Wigner operator. We first define the density fluctuation function, or the phase space dynamical structure function, as

$$S(11', t - t') = \frac{1}{2} \langle [\delta f(1, t), \delta f(1', t')]_+ \rangle, \quad (2.18)$$

where $[\delta f(1, t), \delta f(1', t')]_+$ denotes the anticommutator and $\delta f = \delta f(1) - \langle f(1) \rangle$. If we Fourier transform (2.18) with respect to space and time, we obtain

$$\begin{aligned} S(k, pp'; \omega) &= \int d^3(r_1 - r'_1) e^{-i\vec{k} \cdot (\vec{r}_1 - \vec{r}'_1)} \\ &\quad \times \int_{-\infty}^{\infty} dt_1 dt'_1 e^{i\omega(t_1 - t'_1)} \\ &\quad \times S(11', t_1 - t'_1). \end{aligned} \quad (2.19)$$

The usual Van Hove function³ $S(k\omega)$ determined by

inelastic-neutron-scattering experiments (neglecting spin) is related to $S(k, pp', \omega)$ by

$$nS(k, \omega) = \int d^3p d^3p' S(k, pp'; \omega) \frac{2}{1 + e^{-\beta\hbar\omega}} \quad (2.20)$$

(The integral over p and p' of $S(k, pp'; \omega)$ without the statistical factor $2/(1 + e^{-\beta\hbar\omega})$ is the "symmetrized" scattering function.)

The second function of interest is the Kubo response function, which describes the response of the system to the turning off at time t of a weak perturbation. We define it by

$$\mathcal{L}(11', t - t') = \beta^{-1} \int_0^{\beta} d\lambda \langle \delta f(1', t') \delta f_\lambda(1, t) \rangle, \quad (2.21)$$

where

$$f_\lambda = e^{-\lambda H} f e^{+\lambda H}. \quad (2.22)$$

The relationship of the definition (2.21) to other equivalent definitions of \mathcal{L} is discussed by Kadanoff and Martin¹⁸ and Mazenko.¹¹ Finally, we can also define the commutator correlation function or the generalized susceptibility as

$$\chi(11', t - t') = \frac{1}{2\hbar} \langle [\delta f(1, t), \delta f(1', t')] \rangle. \quad (2.23)$$

It is well known that these three functions are not independent; the relations between them are conveniently expressed in terms of their temporal Fourier and Laplace transforms. The temporal Fourier transform of \mathcal{L} is defined as

$$\mathcal{L}(k, pp'; \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \mathcal{L}(k, pp', t), \quad (2.24)$$

and the temporal Laplace transform is given by

$$\mathcal{L}(k, pp'; z) = -i \int_0^{\infty} dt e^{izt} \mathcal{L}(k, pp', t). \quad (2.25)$$

The variable ω is a real frequency, and the variable z is a complex frequency with $\text{Im} z > 0$. The function $S(k, pp', z)$ is defined as in Eq. (2.25). The Laplace and Fourier transforms are related by

$$\mathcal{L}(k, pp'; \omega) = -2 \text{Im} \mathcal{L}(z = \omega + i0^+), \quad (2.26)$$

or conversely

$$\mathcal{L}(k, pp'; z) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mathcal{L}(k, pp'; \omega)}{z - \omega}. \quad (2.27)$$

To correspond with common notation, we define the temporal transforms of $\chi(11', t)$ as

$$\chi''(k, pp'; \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \chi(k, pp', t), \quad (2.28)$$

and

$$\chi(k, pp'; z) = -2i \int_0^{\infty} dt e^{izt} \chi(k, pp', t), \quad (2.29)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\chi''(k, pp'; \omega)}{z - \omega}. \quad (2.30)$$

The functions S and χ'' are related by the fluctuation-dissipation theorem

$$S(k, pp'; \omega) = \hbar \coth(\frac{1}{2}\beta\hbar\omega) \chi''(k, pp'; \omega), \quad (2.31)$$

and \mathcal{L} and χ'' are related by

$$\frac{1}{2}(\beta\omega)\mathcal{L}(k, pp'; \omega) = \chi''(k, pp'; \omega). \quad (2.32)$$

In the classical limit ($\hbar \rightarrow 0$), the distinction between \mathcal{L} and S vanishes. We can combine (2.27), (2.30), and (2.32) to obtain

$$\mathcal{L}(k, pp'; z) = (\beta z)^{-1} [\chi(k, pp'; z) - \chi(k, pp'; z=0)]. \quad (2.33)$$

The exact static properties of the system enter through the initial value of the time-dependent correlation functions. Momentum-dependent static correlation functions will be denoted by a tilde, e.g., $\tilde{\mathcal{L}}(k, pp') \equiv \mathcal{L}(k, pp'; t=0)$. The relation

$$\tilde{\mathcal{L}}(k, pp') = -\beta^{-1} \chi(k, pp'; z=0) \quad (2.34)$$

will be useful in the following and is obtained in the same manner as (2.33).

We shall require in the following the three-point correlation functions

$$\mathcal{L}(12; 3, t-t') = \beta^{-1} \int_0^\beta d\lambda \langle \delta f(3, t') \delta f_\lambda(12, t) \rangle, \quad (2.35)$$

$$\chi(12; 3, t-t') = \frac{1}{2\hbar} \langle [\delta f(12, t), \delta f(3, t')] \rangle, \quad (2.36)$$

and the four-point functions

$$\mathcal{L}(12; 34, t-t') = \beta^{-1} \int_0^\beta d\lambda \langle \delta f(34, t') \delta f_\lambda(12, t) \rangle, \quad (2.37)$$

$$\chi(12, 34, t-t') = \frac{1}{2\hbar} \langle [\delta f(12, t), \delta f(34, t')] \rangle. \quad (2.38)$$

It will also be convenient to define the inverse of the correlation functions, e.g.,

$$\mathcal{L}^{-1}(1\bar{1}, z) \mathcal{L}(\bar{1}2, z) = \delta(1-2). \quad (2.39)$$

III. QUANTUM KINETIC EQUATION

The dynamics of the system can be expressed in terms of any of the two-point time-dependent correlation functions discussed in Sec. II B. Since these correlation functions are related by (2.31) and (2.32), the choice of correlation function is a matter of convenience. However, the function χ does not satisfy a simple kinetic equation [see (3.20)] and does not have a well-behaved static inverse at small k . Boley and Smith¹² have considered the kinetic equation for S and have discussed a perturbation expansion for its associated memory function in the weak-coupling approxima-

tion. For reasons which will become clear in the following, we prefer to formulate our theory in terms of \mathcal{L} rather than S . The quantity of physical interest $S(k, \omega)$ is then obtained from (2.31) and (2.32).

We obtain the equation of motion of $\mathcal{L}(11'; z)$ by first considering the equation of motion for $\chi(11'; z)$. If we use (2.10) together with the time-translational invariance of the equilibrium ensemble, we write (2.29) as

$$\chi(11'; z) = \frac{1}{2\hbar} \langle [(z+L)^{-1} \delta f(1), \delta f(1')] \rangle. \quad (3.1)$$

The equation of motion of $\chi(z)$ can be found by using the operator identity

$$(z+L)^{-1} = z^{-1} - z^{-1}(z+L)^{-1}L \quad (3.2)$$

to find

$$z\chi(11'; z) = \tilde{\chi}(11') - \frac{1}{2\hbar} \langle [(z+L)^{-1} L \delta f(1), \delta f(1')] \rangle. \quad (3.3)$$

We use (2.11a) to determine the result of the operator L on $\delta f(1)$ in (3.3) and rewrite (3.3) as

$$[z - L_0(1)]\chi(11'; z) - \int d\bar{1} \bar{L}_I(1\bar{1})\chi(1\bar{1}; 1'; z) = \tilde{\chi}(11'), \quad (3.4)$$

where $\chi(11; 1'; z)$ is the temporal Laplace transform of (2.36) and $\tilde{\chi}(11') \equiv \chi(11', t=0)$. The static function $\tilde{\chi}(11')$ can be evaluated explicitly from (2.23) and expressed conveniently in terms of its spatial Fourier transform. We have

$$\tilde{\chi}(k, pp') = \frac{-1}{\hbar} \delta(p-p') \left[f\left(p - \frac{\hbar k}{2}\right) - f\left(p + \frac{\hbar k}{2}\right) \right]. \quad (3.5)$$

From Eq. (3.4), we see that $\chi(11'; z=0) \equiv \chi(11'; 0)$ satisfies

$$-L_0(1)\chi(11'; 0) - \int d\bar{1} \bar{L}_I(1\bar{1})\chi(1\bar{1}; 1'; 0) = \tilde{\chi}(11'). \quad (3.6)$$

We obtain the equation of motion of \mathcal{L} by subtracting (3.6) from (3.4) and use the identity (2.33) to eliminate $\chi(11')$ in favor of $\mathcal{L}(11')$. [An identity analogous to (2.33) holds for the three-point functions $\chi(1\bar{1}; 1'; z)$ and $\mathcal{L}(1\bar{1}; 1'; z)$.] The result is

$$[z - L_0(1)]\mathcal{L}(11'; z) - \int d\bar{1} \bar{L}_I(1\bar{1})\mathcal{L}(1\bar{1}; 1'; z) = -\beta^{-1}\chi(11'; 0) = \mathcal{L}(11'). \quad (3.7)$$

$\chi(11'; 0)$ serves as the initial condition for the $\mathcal{L}(11'; z)$ in (3.7) and is determined by (3.6). For a noninteracting system, the spatial Fourier trans-

form $\chi(k, pp'; 0)$ is found from (3.5), (3.6), and (2.15) to be

$$\chi_0(k, pp'; 0) = \frac{-\delta(p - p')}{\hbar \vec{k} \cdot \vec{p}/m} \times [n^0(p - \hbar k/2) - n^0(p + \hbar k/2)] . \quad (3.8)$$

The small k limit of (3.8) is

$$\chi_0(k=0, pp'; 0) = \delta(p - p') \frac{\partial n^0(\epsilon)}{\partial \epsilon} , \quad (3.9)$$

where $\epsilon = p^2/2m$. The derivative $\partial n^0(\epsilon)/\partial \epsilon$ becomes the familiar δ function of energy at the Fermi surface in the limit of zero temperature.

The equation of motion (3.7) relates the two-point function $\mathcal{L}(11'; z)$ to the three-point function $\mathcal{L}(11; 1'; z)$ and is the first of a hierarchy of correlation function equations. This hierarchy is similar in form to the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy¹⁹ of equations familiar from classical nonequilibrium statistical mechanics. We expect that since \mathcal{L} is the linear response to an adiabatic external perturbation, \mathcal{L} has a resonant structure. It is now well known that it is desirable to consider instead a function that describes the location and width of the resonant structure. Since the location and width are more slowly varying functions of wave vector and frequency, they should be more amenable to approximation than the correlation function itself. We expect that the phase-space memory function or generalized collision integral ϕ defined by the generalized kinetic equation

$$[z - L_0(1)] \mathcal{L}(11'; z) - \int d\bar{2} \phi(1\bar{2}; z) \mathcal{L}(\bar{2}1'; z) = \bar{\mathcal{L}}(11') \quad (3.10)$$

controls the location and width of the resonant structure of \mathcal{L} . The memory function has the physical interpretation of a nonlocal, non-Markovian external source modifying the propagation of a single free-streaming particle described by $z - L_0(1)$.

The major problem in kinetic theory then becomes the development of useful calculational procedures for ϕ . A comparison of (3.10) and (3.7) shows that the memory function satisfies

$$\int d\bar{1} \phi(1\bar{1}; z) \mathcal{L}(\bar{1}1'; z) = \int d\bar{1} L_I(1\bar{1}) \mathcal{L}(1\bar{1}; 1'; z). \quad (3.11)$$

Equation (3.11) is not a convenient expression for ϕ since it involves the unknown function $\mathcal{L}(11'; z)$, and since the right-hand side is not symmetrical in the interaction operator L_I . A more convenient form can be found by using the adjoint equation to (3.7)

$$[z + L_0(1')] \mathcal{L}(11'; z) + \int d\bar{1} L_I(1\bar{1}) \mathcal{L}(1; 1'\bar{1}; z) = -\beta^{-1} \chi(11'; 0) , \quad (3.12)$$

and the equation of motion for the three-point function $\mathcal{L}(11'; 2; z)$

$$[z + L_0(2)] \mathcal{L}(11'; 2; z) + \int d\bar{2} L_I(2\bar{2}) \mathcal{L}(11'; 2\bar{2}; z) = -\beta^{-1} \chi(11'; 2; 0) . \quad (3.13)$$

We apply the operator $[z + L_0(1')]$ to both sides of (3.11) and use (3.12) and (3.13) to obtain

$$\begin{aligned} \phi(1\bar{1}; z) [-\beta^{-1} \chi(\bar{1}1'; 0) - L_I(1'\bar{2}) \mathcal{L}(\bar{1}; 1'\bar{2}; z)] \\ = L_I(1\bar{1}) [-\beta^{-1} \chi(1\bar{1}; 1'; 0) - L_I(1'\bar{2}) \mathcal{L}(1\bar{1}; 1'\bar{2}; z)] . \end{aligned} \quad (3.14)$$

(Repeated barred indices such as $\bar{1}$ in (3.14) imply an integration²⁰ over that variable.) An inspection of (3.14) shows that ϕ separates naturally into two parts

$$\begin{aligned} \phi(1\bar{1}; z) \chi(\bar{1}1'; 0) = \phi^{(s)}(1\bar{1}) \chi(\bar{1}1'; 0) \\ + \phi^{(c)}(1\bar{1}; z) \chi(\bar{1}1'; 0) , \end{aligned} \quad (3.15)$$

where the static (z -independent) part is given by

$$\phi^{(s)}(1\bar{1}) \chi(\bar{1}1'; 0) = L_I(1\bar{1}) \chi(1\bar{1}; 1'; 0) , \quad (3.16)$$

and the collisional part is given by

$$\begin{aligned} \phi^{(c)}(1\bar{1}) \chi(\bar{1}1'; 0) \\ = - \int d\bar{1} d\bar{2} L_I(1\bar{1}) L_I(1'\bar{2}) \\ \times [\mathcal{L}(1\bar{1}; 1'\bar{2}; z) \\ - \mathcal{L}(1\bar{1}; \bar{3}; z) \mathcal{L}^{-1}(\bar{3}\bar{4}; z) \mathcal{L}(\bar{4}; 1'\bar{2}; z)] . \end{aligned} \quad (3.17)$$

The formal expressions (3.16) and (3.17) for $\phi^{(s)}$ and $\phi^{(c)}$ are identical in structure to the analogous expressions for classical systems. However (3.16) and (3.17) are integral equations for $\phi^{(s)}$ and $\phi^{(c)}$, respectively, because of the presence of $\chi(11'; 0)$ (and the implied integration) on the left-hand side. It is one of the major simplifications of classical systems that the momentum dependence of static correlation functions such as $\bar{\mathcal{L}}(11') = \beta^{-1} \chi(11'; 0)$ is known, and thus $\phi^{(s)}$ and $\phi^{(c)}$ can be solved for explicitly.

Note that the form of $\phi^{(s)}$ as given in (3.16) occurs naturally in (3.6) for $\chi(11'; 0)$. We can rewrite our generalized kinetic equation (3.10) in terms of the static and dynamic parts of the memory function as

$$\begin{aligned} & \left(z - \frac{\vec{k} \cdot \vec{p}}{m} \right) \mathcal{L}(k, pp'; z) - \phi^{(s)}(k, p\bar{p}) \mathcal{L}(k, \bar{p}p'; z) \\ & - \phi^{(c)}(k, p\bar{p}; z) \mathcal{L}(k, \bar{p}p'; z) \\ & = -\beta^{-1} \chi(k, pp'; 0), \quad (3.18) \end{aligned}$$

where only the static part of the memory function appears in the boundary condition for $\chi(k, pp'; 0)$

$$\begin{aligned} & \frac{\vec{k} \cdot \vec{p}}{m} \chi(k, pp'; 0) + \phi^{(s)}(k, p\bar{p}) \chi(k, \bar{p}p'; 0) \\ & = \frac{1}{\hbar} \delta(p - p') [f(p - \hbar k/2) - f(p + \hbar k/2)]. \quad (3.19) \end{aligned}$$

The first term in (3.18) represents free particle streaming. The physical interpretation of $\phi^{(s)}$ and $\phi^{(c)}$ in (3.18) is the same as in classical kinetic theory, that is, $\phi^{(s)}$ corresponds to a mean-field or generalized random-phase term and $\phi^{(c)}$ represents the dynamical effects of collisions.

The form of (3.19) is convenient since it relates the two-point static functions $\chi(k, pp'; 0)$ and $\phi^{(s)}(k, pp')$ to the one-particle function $f(p)$ which has a simple physical interpretation and which in principle can be measured experimentally. Boley and Smith¹² have shown that $S(k, pp'; z)$ also satisfies a generalized kinetic equation of the same form as (3.18). However, the analogous condition to (3.19) involves the static part of the memory function of $S(k, pp'; z)$ and the two-point function $\bar{S}(k, pp') \equiv S(k, pp', t=0)$. Little is known from theoretical considerations about the behavior of $\bar{S}(k, pp')$, nor can it be measured experimentally. We shall see in Sec. IV that the relatively simple form of (3.19) will allow us to adopt a simple phenomenological model for $\phi^{(s)}$ and χ .

In contrast to \mathcal{L} and S , the generalized susceptibility χ does not satisfy an equation of the form (3.18). If we substitute the relation (2.33) into (3.18), we find that χ satisfies the equation

$$\begin{aligned} & \left(z - \frac{\vec{k} \cdot \vec{p}}{m} \right) \chi(k, pp'; z) - \phi^{(s)}(k, p\bar{p}) \chi(k, \bar{p}p'; z) \\ & - \phi^{(c)}(k, p\bar{p}; z) [\chi(k, \bar{p}p'; z) - \chi(k, \bar{p}p'; 0)] \\ & = \frac{-1}{\hbar} \delta(p - p') [f(p - \hbar k/2) - f(p + \hbar k/2)]. \quad (3.20) \end{aligned}$$

It is clear that the form of (3.20) is less convenient than (3.19).

Our problem of calculating $\mathcal{L}(z)$ has now been shifted to calculating $\phi^{(s)}$, $\phi^{(c)}$, and $f(p)$, from which \mathcal{L} and $S(k, \omega)$ can be found. The motivation for this reformulation is the expectation that the results for \mathcal{L} and S should be relatively insensitive

to the approximations made for $\phi^{(s)}$, $\phi^{(c)}$, and $f(p)$. For example, if we set $\phi^{(s)} = \phi^{(c)} = 0$ and replace $f(p)$ with its free-particle value, we easily obtain the free-particle forms for $\mathcal{L}(z)$ and $\chi(z=0)$ [see (3.8)]. In general, the k and z dependence of the memory function should be slowly varying and can thus be simply parametrized. In the following section, we develop the correspondence between our formalism and the existing phenomenological theory of normal Fermi liquids in order to identify the important parameters characterizing $\phi^{(s)}$ and $\phi^{(c)}$, and to appraise the usefulness of the formalism. The next step in a complete theory is to carry out microscopic calculations for $\phi^{(s)}$ and $\phi^{(c)}$. It is possible to perform a direct perturbation expansion in the potential for $\phi^{(s)}$, $\phi^{(c)}$, and $f(p)$ similar to that given by Boley and Smith.¹² However for realistic potentials, i.e., adequate to represent helium, such a perturbation expansion is not adequate. We know that in classical kinetic theory, density expansions of $\phi^{(c)}$ provide a satisfactory alternative to perturbation theory, and that it is possible to obtain at least the qualitative behavior for strongly interacting systems. Unfortunately, a simple density expansion for $\phi^{(c)}$ in the quantum case does not conveniently preserve the statistics. We hope to discuss in future work a new microscopic method which reduces to the usual density expansion at high temperatures and which preserves the quantum statistics at low temperatures. Such a method should be capable of giving realistic approximations for $\phi^{(s)}$, $\phi^{(c)}$, and $f(p)$ for strongly interacting systems.

IV. THE PHENOMENOLOGICAL THEORY

In lieu of a sophisticated microscopic analysis of the properties of the memory function of liquid ³He, we present a simple phenomenological model of the memory function which is applicable to liquid ³He at low temperatures. As pointed out in Sec. III, one of the advantages of the present formulation in terms of \mathcal{L} is the comparative ease with which it is possible to relate the formal expressions to those of phenomenological theories. Our approach is to relate the formal expressions (3.16) and (3.17) for $\phi^{(s)}$ and $\phi^{(c)}$, respectively, to the analogous small k and ω results of Landau's theory of a normal Fermi liquid. Higher k and ω effects in ϕ are included by using the method of kinetic modeling²² developed in classical kinetic theory.

We first investigate $\phi^{(s)}$ and the associated function $\chi(z=0)$ in the low temperature, small wave-number limit. In this limit (and in general the limit of small frequency), normal liquid ³He is represented in the Landau theory as a dilute gas of

quasiparticles with an effective mass m^* different from the "bare" mass m of an individual ^3He atom. The equilibrium one-particle distribution function $f(p)$ takes the form

$$f(p) = [e^{\beta(\epsilon(p) - \mu)} + 1]^{-1}, \quad (4.1)$$

with $\epsilon(p) = p^2/2m^*$. An inspection of (3.19) shows that for the particles to acquire a mass m^* , $\phi^{(s)}$ must be of the form (setting $\hbar = 1$ for convenience)

$$\phi_0^{(s)} = \delta(p - p') \frac{\vec{k} \cdot \vec{p}}{m'} \quad (4.2)$$

in the limit of small k . It is easy to see that m^* is related to the constant m' by

$$\frac{1}{m^*} = \frac{1}{m} + \frac{1}{m'}. \quad (4.3)$$

An analysis²¹ of the formal expression (3.16) for $\phi^{(s)}$ shows that $\phi^{(s)}$ does have the form (4.2) for small k . A discussion of the microscopic calculation of m' will be postponed to future work. We take m^* as an input parameter and rewrite (3.19) in terms of m^* , the approximate one-particle distribution function given by (4.1), and $\phi^{(s)}$. The result is

$$\chi(k, pp'; 0) + \bar{\phi}^{(s)}(k, p\bar{p})\chi(k, \bar{p}p'; 0) = F_k(p)\delta(p - p'), \quad (4.4)$$

where

$$F_k(p) = \frac{m^*}{\vec{k} \cdot \vec{p}} [f(p - \frac{1}{2}k) - f(p + \frac{1}{2}k)], \quad (4.5)$$

and

$$\bar{\phi}^{(s)}(k, pp') = \frac{m^*}{\vec{k} \cdot \vec{p}} [\phi^{(s)}(k, pp') - \phi_0^{(s)}(k, pp')]. \quad (4.6)$$

A determination of the form of $\phi^{(s)}$ also determines the form of $\chi(k, pp'; 0)$. In order to see explicitly the relation between $\phi^{(s)}$ and χ , and to understand the relation of (4.4) to the Landau theory, we consider the $k=0$ limit of $\phi^{(s)}$ and χ and write

$$\bar{\phi}^{(s)}(k=0, pp') = N(0)^{-1} \sum_{lm} F_l F_0(p) Y_{lm}(\hat{p}) Y_{lm}(\hat{p}'), \quad (4.7)$$

and

$$\chi(k=0, pp'; 0) = \sum_{lm} \chi_l F_0(p) \times F_0(p') Y_{lm}(\hat{p}) Y_{lm}(\hat{p}'). \quad (4.8)$$

The Y_{lm} are the usual spherical harmonics; $N(0)$ is the density of states at the Fermi surface and is given by

$$N(0) = \sum_p F_0(p).$$

The spin $\frac{1}{2}$ of the ^3He atoms is taken into account by interpreting \sum_p to implicitly include a sum over spin. At $T=0$, $F_0(p) = -\partial f(\epsilon)/\partial \epsilon = \delta(p^2/2m - \mu)$, and $N(0) = m^* p_F / \pi^2$. We substitute (4.7) and (4.8) into the $k=0$ limit of (4.4) and use the addition theorem for spherical harmonics to find

$$\chi_l = N(0) [1 + F_l / (2l + 1)]^{-1}. \quad (4.9)$$

The relation (4.9) becomes familiar if we identify F_l with the expansion coefficients $F_l^{(s)}$ of $f_{pp'}$, where $f_{pp'}$ is the quasiparticle interaction energy⁴ of the Landau theory.

Since the spin-orbit coupling in liquid ^3He is extremely small, we can develop our kinetic theory for the density and spin parts of the various correlation functions separately, and in an obviously analogous way. In particular, we can obtain equations for the spin part of the generalized susceptibility similar to (4.4) and (4.9). In order to avoid repetition we will continue to develop the theory for the density part only, and indicate only some of the results for the spin part. We shall see that the density part is more interesting, since it exhibits a collective mode.

We have seen that in the small k limit, $\bar{\phi}^{(s)}$ and $\chi(0)$ reduce to a simple form in terms of the Landau parameters. We wish to investigate the form of $\bar{\phi}^{(s)}$ for arbitrary k by considering its matrix representation.²² In order to gain insight into the properties of its matrix elements, we consider first the parametrization of $\phi^{(c)}$ in the classical case. We introduce a set of basis functions $\psi_i(p)$ that satisfy the orthogonality and completeness conditions

$$\int d^3p W(p) \psi_i(p) \psi_j(p) = \delta_{ij}, \quad (4.10)$$

$$\sum_{i=1}^{\infty} \psi_i(p) \psi_i(p') W(p) = \delta(p - p'), \quad (4.11)$$

where $W(p)$ is a weight function. In the classical case $W(p)$ is taken to be the Boltzmann distribution function $(2\pi\beta/m)^{-3/2} e^{-\beta p^2/2m}$. The memory function $\phi^{(c)}$ is written in the matrix representation

$$\phi^{(c)}(k, pp'; z) = \sum_{ij} \psi_i(p) \psi_j(p') W(p) \phi_{ij}^{(c)}(k, z), \quad (4.12)$$

where the matrix elements $\phi_{ij}^{(c)}(k, z)$ are defined by

$$\phi_{ij}^{(c)}(k, z) = \int d^3p d^3p' W(p) \times \psi_i(p) \psi_j(p') \phi^{(c)}(k, pp'; z). \quad (4.13)$$

The functions $\psi_i(p)$ are ordered such that the first five functions are proportional to the five hydro-

dynamical functions corresponding to conservation of particles, momentum, and energy. The matrix elements $\phi_{ij}^{(c)}$ for $i, j < 5$ are qualitatively different from the remaining matrix elements and vanish as k and z go to zero. If i and j are not in the hydrodynamical set, $\phi_{ij}^{(c)}(k, z)$ is approximately diagonal in i and j and $\phi_{ii}^{(c)}(k, z)$ varies slowly with increasing i . These properties of $\phi_{ij}^{(c)}$ are included in the systematic approximation scheme of kinetic modeling.

The static part of the memory function can also be expanded in terms of its matrix elements. However in the classical case⁶ the form of $\phi^{(s)}$ can be solved for explicitly, $\phi^{(s)}(k, pp') = -(\vec{k} \cdot \vec{p}/m) \times C_D(k)W(p)$, and only one matrix element of $\phi^{(s)}$ enters. [$C_D(k)$ is the direct correlation function.] Since we cannot explicitly solve the integral equation (3.16) for $\phi^{(s)}$ in the quantum case, we write

$$\bar{\phi}^{(s)}(k, pp') = \sum_{ij} \psi_i(p) \psi_j(p') \bar{\phi}_{ij}^{(s)}(k) F_k(p). \quad (4.14)$$

We have chosen the weight function in (4.14) to be the Fermi factor $F_k(p)$ given by (4.5). The functions $\psi_i(p)$ now depend weakly on k , satisfy (4.10) and (4.11) with $W(p) = F_k(p)$, and reduce to the hydrodynamic functions in the limit $k \rightarrow 0$. The hydrodynamic functions are

$$\begin{aligned} \phi_1 &= 1, \quad \phi_2 = p_z = p \cos \theta \\ \phi_{3,4} &= p_{x,y}, \quad \phi_5 = p^2/2m^* - \mu, \end{aligned} \quad (4.15a)$$

and those corresponding to their currents are

$$\begin{aligned} \phi_{6,7,8} &= \left(\frac{p^2}{2m^*} - \mu \right) p_{x,y,z}, \\ \phi_9 &= p_z^2, \quad \phi_{10} = p_x^2 - p_y^2, \\ \phi_{11} &= p_x p_y, \quad \phi_{12,13} = p_{x,y} p_z. \end{aligned} \quad (4.15b)$$

The z axis is taken along k . The functions ϕ_i are not all mutually orthogonal, but each member of the longitudinal subset $\{\phi_1, \phi_2, \phi_5, \phi_8, \phi_9, \dots\}$ is orthogonal to each of the remaining functions. Because of the spherical symmetry of the system, only the matrix elements corresponding to the longitudinal subset will enter directly into (4.14). The functions $\{\phi_1, \phi_2, \phi_5, \phi_8, \phi_9, \dots\}$ are not orthogonal to each other but in the limit $k \rightarrow 0$ the functions $\{\phi_1, \phi_2, \phi_5, \phi_8, 3\phi_9 - \phi_1, \dots\}$ are. The explicit form of the properly normalized basis functions ψ_i through $i = 5$ is given in the Appendix. Only the explicit forms of ψ_1 and ψ_2 are of interest here. We have

$$\psi_1(p) = [N(0)N_1(x)]^{-1/2}, \quad (4.16)$$

$$\psi_2(p) = [3/N(0)]^{1/2}(p/p_F) \cos \theta, \quad (4.17)$$

where

$$N_1(x) = \frac{1}{2} \left(1 + \frac{1-x^2}{2x} \ln \left| \frac{1+x}{1-x} \right| \right), \quad (4.18)$$

and $x = k/2p_F$. It is easy to see that up to a normalization factor ψ_1 and ψ_2 correspond to the $l = 0$ and $l = 1$ Legendre polynomials in (4.7).

If we also write χ as in (4.14), we can solve for the matrix elements χ_{ij} in terms of $\bar{\phi}_{ij}^{(s)}$ by using (4.4). We construct an approximation of order N by truncating the basis set $\{\psi_i\}$ at $i = N$, multiply (4.4) by $\psi_i(p)\psi_j(p')$, use the orthonormality property (4.10), and obtain a set of $2N$ equations for $\bar{\phi}_{ij}^{(s)}$ and χ_{ij} . As an example, we take

$$\begin{aligned} \bar{\phi}^{(s)}(k, pp') &= [\bar{\phi}_{11}^{(s)}(k)\psi_1(p)\psi_1(p') \\ &+ \bar{\phi}_{22}^{(s)}(k)\psi_2(p)\psi_2(p')] F_k(p) \end{aligned} \quad (4.19)$$

and obtain the following nonzero matrix elements

$$\begin{aligned} \chi_{11}(k) &= N(0)[1 + \bar{\phi}_{11}^{(s)}(k)]^{-1}, \\ \chi_{22}(k) &= N(0)[1 + \bar{\phi}_{22}^{(s)}(k)]^{-1}. \end{aligned} \quad (4.20)$$

The approximate form (4.19) for $\bar{\phi}^{(s)}$ is applicable to all k and is in terms of two as yet undetermined parameters. Although it is possible in principle to determine $\bar{\phi}_{11}^{(s)}$ and $\bar{\phi}_{22}^{(s)}$ microscopically, we shall discuss in the following how they can be determined from the first two sum rules for $S(k, \omega)$. In order to make contact with the Landau theory, we rewrite (4.19) as

$$\bar{\phi}^{(s)}(k, pp') = [\alpha(k) + \beta(k)(\vec{p} \cdot \hat{k})(\vec{p}' \cdot \hat{k})/p_F^2] F_k(p), \quad (4.21)$$

where p_F is the Fermi momentum and \hat{k} denotes a unit vector. In the limit $k \rightarrow 0$, α and β can be identified with the Landau parameters [see (4.7) and (4.9)].

$$\alpha(0) = F_0^s; \quad \beta(0) = F_1^s. \quad (4.22)$$

We now turn to the modeling of the collisional part of the memory function $\phi^{(c)}$ and write $\phi^{(c)}$ as in (4.12) with $W(p) = F_k(p)$. We can model $\phi^{(c)}$ by incorporating what we know from the discussion of kinetic modeling in the classical case and from a discussion of the Landau limit $k \rightarrow 0$, $\omega/v_F \rightarrow 0$, $T \rightarrow 0$ of the collision kernel (v_F is the Fermi velocity). A realistic approximation for $\phi^{(c)}$ must be consistent with the conservation laws, i.e.,

$$\sum_p \phi_i(p) \phi^{(c)}(k=0, pp'; z=0) = 0, \quad (4.23)$$

for $i = 1$ through 5. We consider the limit k and z small so that the diagonal matrix elements for $i \leq 5$ can be neglected and assume that as in the classical case the off-diagonal matrix elements are small compared to the diagonal matrix elements. If we also approximate the diagonal matrix elements for $i > 5$ by $-i\tau^{-1}$, we can use (4.11) to write

$\phi^{(c)}$ for small k and z as

$$\phi^{(c)}(k, pp'; z) = -i\tau^{-1} [\delta(p - p') - \sum_{i=1}^5 \psi_i(p) \psi_i(p') F_k(p)] . \quad (4.24)$$

Equation (4.24) is the simplest approximation to $\phi^{(c)}$ that we can obtain by analogy to the classical case which satisfies (4.23). Note that the form of (4.24) differs from what would have been found from simple truncation of (4.12). The nonhydrodynamic ($i > 5$) diagonal matrix elements have been written with the factor of $-i$ in order that the transport coefficients be real and positive.

We can gain further insight into the approximate form of $\phi^{(c)}$ from the Landau²³ form of the collision kernel. It is easy to see that (4.24) is identical to what would have been found by calculating the matrix elements of the Landau collision kernel directly, if we identify τ with the matrix element²⁴ associated with the shear viscosity η . The simple relation between η and τ obtained in Ref. 4, within the Landau theory, is

$$\eta = \frac{1}{5} nm^* v_F^2 \tau . \quad (4.25)$$

The relaxation time τ is proportional to T^{-2} and can be expressed in terms of the Landau parameters. We adopt an alternative approach and use (4.25) to determine τ from the experimental²⁵ values of η .

A study of the Landau limit of $\phi^{(c)}$ also implies that the terms $i=3, 4, 5$ can be neglected in (4.24). The term $i=5$ associated with energy conservation can be neglected since at low temperature the

thermal diffusivity mode makes a negligible¹⁸ contribution to $S(k, \omega)$ in the hydrodynamic limit. The transverse momentum terms ψ_3 and ψ_4 can also be neglected since they will not enter into the calculation of a longitudinal correlation function.

Because of the hard-core nature of the ^3He interparticle interaction, we expect that the k and z dependence of $\phi^{(c)}$ is small. We thus take our low-temperature approximation for $\phi^{(c)}$ to be

$$\phi^{(c)}(k, pp'; z) = -i\tau^{-1} [\delta(p - p') - \sum_{i=1,2} \psi_i(p) \psi_i(p') F_k(p)] , \quad (4.26)$$

where τ is given by (4.25).

We now turn our attention to the dynamical equation (3.18) for \mathcal{L} which we rewrite in terms of $\bar{\phi}^{(s)}$ and m^* as

$$\left(z - \frac{\vec{k} \cdot \vec{p}}{m^*} \right) \mathcal{L}(k, pp'; z) - \frac{\vec{k} \cdot \vec{p}}{m^*} \bar{\phi}^{(s)}(k, p\bar{p}) \mathcal{L}(k, \bar{p}p'; z) - \phi^{(c)}(k, p\bar{p}; z) \mathcal{L}(k, \bar{p}p'; z) = -\beta^{-1} \chi(k, pp'; 0) . \quad (4.27)$$

The solution of (4.27) with (4.21) for $\bar{\phi}^{(s)}$ and (4.26) for $\phi^{(c)}$ can be found by dividing both sides by $(z + \vec{k} \cdot \vec{p}/m^* + i\tau^{-1})$, multiplying both sides by $\psi_1(p)\psi_1(p')$ or $\psi_1(p)\psi_2(p')$, and then integrating over p and p' . We obtain a set of two equations and two unknowns which we solve for

$$n\mathcal{L}(k, z) = \int d^3p d^3p' \mathcal{L}(k, pp'; z) , \quad (4.28)$$

where $\mathcal{L}(k, z)$ is essentially the 1, 1 matrix element of $\mathcal{L}(k, pp'; z)$. The result is

$$\mathcal{L}(k, z) = \left(\frac{-\beta^{-1} N(0)}{1 + \alpha N_1} \right) \frac{[\Lambda'_{11}(1 - \beta\Lambda_{22} - i\tau^{-1}\Lambda'_{22}) + \Lambda'_{12}(\beta\Lambda_{12} + i\tau^{-1}\Lambda'_{12})]}{[(1 - \alpha\Lambda_{11} - i\tau^{-1}N_1^{-1}\Lambda'_{11})(1 - \beta\Lambda_{22} - i\tau^{-1}\Lambda'_{22}) - (\beta\Lambda_{12} + i\tau^{-1}\Lambda'_{12})(\alpha\Lambda_{12} + i\tau^{-1}N_1^{-1}\Lambda'_{12})]} . \quad (4.29)$$

The functions Λ_{ij} and Λ'_{ij} are given by

$$\Lambda_{ij}(k, z) = \int d^3p \frac{\psi_i(p) \psi_j(p) (N_i N_j)^{1/2} F_k(p) \vec{k} \cdot \vec{p}/m^*}{\bar{z} - \vec{k} \cdot \vec{p}/m^*} , \quad (4.30)$$

and

$$\Lambda'_{ij}(k, z) = \int d^3p \frac{\psi_i(p) \psi_j(p) (N_i N_j)^{1/2} F_k(p)}{\bar{z} - \vec{k} \cdot \vec{p}/m^*} , \quad (4.31)$$

where N_1 is given by (4.18), $N_2 = \frac{1}{3}$, and $\bar{z} = z + i\tau^{-1}$. It is easy to express Λ_{ij} and Λ'_{ij} in terms of Λ_{11} which can be evaluated analytically at $T=0$. The result is (τ evaluated at $T \neq 0$)

$$\Lambda_{11}(k, z) = -\frac{1}{2} + \frac{1}{4k} \left[p_F - \frac{(\bar{z} + k^2/2m^*)^2}{k^2 p_F/m^*} \right] \ln \left(\frac{\bar{z} + k^2/2m^* - kp_F/m^*}{\bar{z} + k^2/2m^* + kp_F/m^*} \right) - \frac{1}{4k} \left[p_F - \frac{(\bar{z} - k^2/2m^*)^2}{k^2 p_F/m^*} \right] \ln \left(\frac{\bar{z} - k^2/2m^* - kp_F/m^*}{\bar{z} - k^2/2m^* + kp_F/m^*} \right) , \quad (4.32)$$

where the symbol In stands for the complex log function and the principal value is understood.

The desired result for $S(k, \omega)$ in terms of the parameters $\alpha(k)$ and $\beta(k)$ can be obtained from (2.20), (2.26), (2.31), (2.32), and (4.32). The functions $\alpha(k)$ and $\beta(k)$ can be specified by requiring that $S(k, \omega)$ satisfy the sum rules

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(k, \omega) = S(k), \quad (4.33)$$

and

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega S(k, \omega) = \frac{k^2}{2m}, \quad (4.34)$$

where $S(k)$ is defined by (2.17). We use the measured values of $S(k)$ at $T \sim 0.4$ K in the range $0.15 \leq k \leq 2.1 \text{ \AA}^{-1}$, perform the integrals over ω in (4.33) and (4.34) numerically, and determine α and β so that the two sum rules are satisfied. The results for α and β are shown in Fig. 1. Since the measured $S(k)$ is a slowly varying function of T in the range of k of interest, the values of α and β shown in Fig. 1 are essentially at $T=0$. Note that $\beta(k)$ is a slowly varying function of k . If we extrapolate the values of α and β determined by the above procedure to $k=0$, we see that α and β appear to approach their limiting values F_0^s and F_1^s , respectively, given by the Landau theory [see (4.22)]. This agreement with the measured²⁵ values, $F_0^s = 10.07$ and $F_1^s = 6.04$, determined from the compressibility and the specific heat, implies that our dynamical approximations are consistent with the statics of the system.

If we use the values for α and β shown in Fig. 1, our form for $S(k, \omega)$, the density fluctuation part of the dynamic-structure function, coincides in the Landau limit of $k \rightarrow 0$, $\omega/v_F k \rightarrow 0$, and $T \rightarrow 0$ with

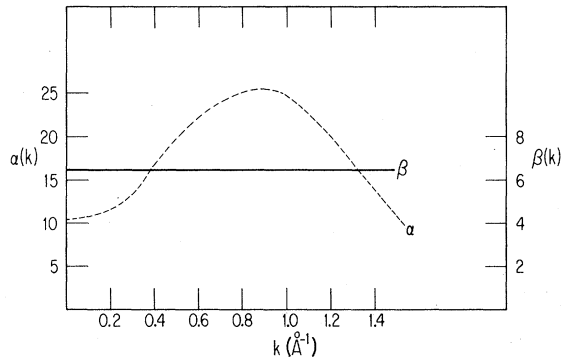


FIG. 1. Parameters α and β as functions of k . Notice that at small k their numerical values coincide with those of the Landau parameters F_0^s and F_1^s .

that found from the phenomenological Landau theory. At higher values of k and ω , the k and ω dependence of $S(k, \omega)$ is more complicated than the Landau theory and arises from the functions Λ_{11} , $\alpha(k)$, and $\beta(k)$. Because we have included the effects of quasiparticle collisions in $\phi^{(e)}$, our expression for $S(k, \omega)$ is different in nature than those given by a generalized random-phase approximation.^{25, 26} The effect of quasiparticle collisions becomes important for nonzero T since the relaxation time τ given by (4.25) varies as T^2 for low T . Temperature effects associated with Λ_{11} are negligible.

The scattering of neutrons from liquid ^3He depends on whether the neutron and the nucleus form an intermediate state which is either a triplet or a singlet state. Since the scattering lengths associated with these two intermediate states are different, the differential cross section for unpolarized neutrons is proportional¹ to

$$S(k, \omega) = S(k, \omega) + (\sigma_i/\sigma_c) S_1(k, \omega), \quad (4.35)$$

where $S_1(k, \omega)$ is the incoherent part (due to spin density fluctuations) of the total dynamic-structure function $S(k, \omega)$; σ_i and σ_c are the incoherent and coherent cross sections, respectively. The analysis of Ref. 2 shows that $\sigma_i \approx 0.25\sigma_c$. We can model $S_1(k, \omega)$ in the same spirit as was done for $S(k, \omega)$. Since $S_1(k, \omega)$ is not expected to show a resonant structure, we adopt a simpler form. We choose a one-parameter model analogous to (4.21) for the static part of the memory function of $S_1(k, \omega)$ with $\alpha(k)$ replaced by $Z(k)$ and $\beta(k)$ replaced by zero. The parameter $Z(k)$ reduces to the Landau value²⁵ $Z_0 = -2.69$ at $k=0$. The spin current density is not conserved so $S_1(k, \omega)$ obeys only the zero moment sum rule analogous to (4.33) with $S(k)$ replaced by $S_1(k)$, the spin static-structure function. $S_1(k)$ was indirectly determined in Ref. 2 and found to be given by the limiting value of unity for $0.8 \text{ \AA}^{-1} \leq k \leq 1.5 \text{ \AA}^{-1}$. This behavior of $S_1(k)$ implies that $Z(k)$ goes to zero in this range of k .

The ω dependence of $S(k, \omega)$ and $S_1(k, \omega)$ at $T = 0.015$ K and $k = 1.4 \text{ \AA}^{-1}$ is given in Fig. 2. There is a small broad peak in $S(k, \omega)$ (the solid line) produced by the single-particle excitations. The smallness of this peak is due to the fact $\alpha(k)$ is large in this range of k . The most striking feature of $S(k, \omega)$ is the sharp peak at higher ω which we identify as the zero-sound resonance. This identification is confirmed by looking at the small k limit of $S(k, \omega)$. The dotted line in Fig. 2 is $S_1(k, \omega)$ which dominates the behavior of $S(k, \omega)$ for $\omega < 1$ meV. The theoretical dispersion relation $\omega_0(k)$ of the zero-sound peak is shown in Fig. 3. It is seen to be linear for small k and then rise to a plateau for larger k .

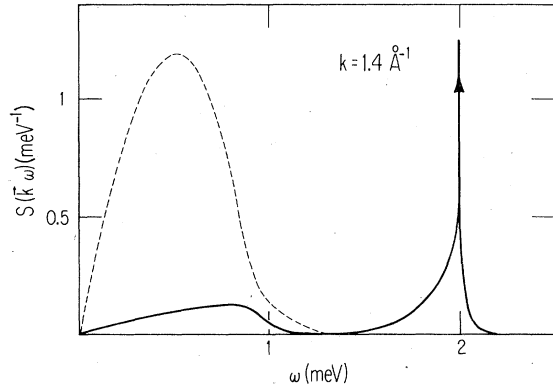


FIG. 2. Total neutron scattering function as given by the theory of this paper for $T = 0.015$ K and $k = 1.4 \text{ \AA}^{-1}$. The solid line is the density part $S(k, \omega)$ and the dashed line is the spin part $S_I(k, \omega)$ multiplied by 0.25. The sharp peak on the right is identified as the zero-sound mode.

V. DISCUSSION

The neutron scattering measurements of the dynamic-structure function of liquid ^3He performed at Argonne² at $T = 0.015$ K are consistent with our theoretical results. Previous experiments¹ at $T = 0.63$ K did not indicate the existence of the zero-sound mode. A generalized random-phase approximation²⁷ based on a modification of the sum rule for the multiparticle-hole part of $S(k, \omega)$ has been proposed to account for the observed temperature dependence of the width of the zero-sound mode. In the kinetic-theory analysis presented here, the temperature dependence of the width of

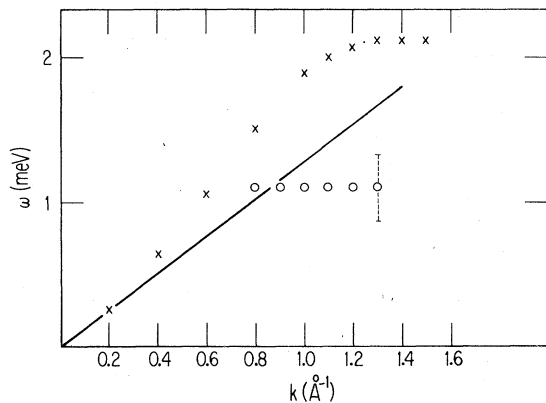


FIG. 3. Dispersion relation for the zero-sound mode. The circles are the experimental points of Ref. 2, and the crosses are our theoretical values. The straight line is an extrapolation of the linear dispersion relation prevailing at very small k .

the zero-sound mode arises from the T^2 dependence of the quasiparticle collision time²⁸; according to our theory the width of the zero-sound mode increases by a factor of approximately 10^3 as T increases from 0.015 to 0.63 K. The temperature-dependent statistical factors in Λ_{11} contribute an additional but small broadening of the zero-sound mode. The quasiparticle collisions are neglected in the random-phase approximation.

Our theoretical results for the zero-sound dispersion curve $\omega_0(k)$ are compared to the experimental results in Fig. 3 and are seen to be in qualitative agreement with experiment but to lie above the experimental curve.²⁹ The qualitative comparison of our theoretical values for $\omega_0(k)$ with the experimental values is reminiscent of the results of the Feynman theory³⁰ for the phonon dispersion curve in liquid ^4He . The reason for the lack of quantitative agreement of the theoretical $\omega_0(k)$ with experiment is our simplified treatment of the collisional part of the memory function for which we have neglected higher-frequency effects and the effects of mode-mode coupling. The latter effect is particularly important since the phase space for zero-sound-zero-sound coupling is appreciable due to the flatness of the zero-sound dispersion curve between 0.8 and 1.4 \AA^{-1} . This mode-mode coupling should give rise to a secondary peak in $S(k, \omega)$ at $\omega \approx 2\omega_0$. There is experimental evidence³¹ for this secondary peak. It is easy to see that the inclusion of extra weight in $S(k, \omega)$ at higher ω would lead to lower values of α and β , and thus $\omega_0(k)$ for a given k . We hope to investigate the effects of mode-mode coupling in future work.

We have seen that it is possible to develop a simple phenomenological model of the dynamic-structure function $S(k, \omega)$ of liquid ^3He based on a generalized kinetic equation for the Kubo response function \mathcal{L} . The model satisfies the first two sum rules for $S(k, \omega)$, is consistent with the conservation laws, includes quasiparticle collisional effects, and yields the temperature dependence of the width of the zero-sound resonance in a natural manner. Results consistent with recent neutron scattering experiments were found.

ACKNOWLEDGMENTS

We would like to thank Dr. Kurt Sköld and Dr. Charles Pelizzari for useful discussions of their experimental results. One of us (H.G.) would like to thank members of the Solid State Division of Argonne National Laboratory for their hospitality during his appointment as a Faculty Research Participant.

APPENDIX

We wish to order the set $\{\psi_i\}$ so that it is manifest that in the Landau limit ($k \rightarrow 0$, $\omega/v_F k \rightarrow 0$, and $T \rightarrow 0$), our expansion of ϕ is of the Landau form (4.7) without the need for rearrangement. In this limit, the Fermi factor $F_k(p)$ requires $p = p_F$, and ψ_1 and ψ_2 coincide up to a normalization constant with the $l=0$ and $l=1$ Legendre polynomials. We construct the $\{\psi_i\}$ so that they satisfy (4.10) and (4.11). The function ψ_5 is written as a linear combination of ϕ_5 and ϕ_9 .

$$\psi_5(p) = \frac{(p/p_F)^2 [\frac{3}{2} \cos^2 \theta - \frac{1}{2} b_5(x)]}{[N(0)N_5(x)]^{1/2}}, \quad (A1)$$

where $b_5(x)$ is chosen so that ψ_5 is orthogonal to ψ_1 . $N_5(x)$ is the normalization constant and $x = k/2p_F$. Note that the k dependence of $\psi_5(p)$ is suppressed, and that ψ_5 is orthogonal to ψ_2 . It is elementary but tedious to verify that

$$b_5(x) = 2[N_1(1-x^2)+1]^{-1}, \quad (A2)$$

and

$$N_5(x) = \frac{1}{4} \left[\frac{1}{3} (3 - b_5)^2 \left[2 \left(\frac{1}{3} x^2 + 1 \right) + N_1(x^2 - 1)^2 \right] + 2(3 - b_5) \left[\left(\frac{1}{3} x^2 + 1 \right) - N_1(x^2 - 1)^2 \right] + 9N_1(1 + 3x^2) \left(\frac{1}{4} x^2 - \frac{1}{5} \right) + \frac{21}{5} - \frac{9}{5} x^2 \right]. \quad (A3)$$

$$b_3(x) = \frac{1 - N_1(1+x^2) + (2 - \frac{10}{9}x^2)c_3}{2N_1(x)}, \quad (A8)$$

$$c_3(x) = \frac{-2N_1(x^2 - 1)^2 + 2(\frac{1}{3}x^2 - 1) + \frac{2}{3}(3 - b_5)[2(\frac{1}{3}x^2 + 1) + N_1(x^2 - 1)^2]}{-2 + \frac{4}{3}x^2 - \frac{46}{75}x^4}, \quad (A9)$$

and

$$N_3(x) = \frac{2}{3} (1 + \frac{1}{3}x^2) + \frac{1}{3}N_1(1-x^2)^2 + N_1(1+b_3)^2 - (1+b_3)(1+N_1)(1-x^2) + 2c_3[(1-x^2)^2 + \frac{16}{25}x^4 + \frac{2}{3}x^2(1-x^2)] - 2c_3(1+b_3)(1+\frac{13}{9}x^2). \quad (A10)$$

Since $b_5(x=0)=1$, it is clear that in the Landau limit ψ_5 reduces essentially to the $l=2$ Legendre polynomial.

We write ψ_4 as a linear combination of ϕ_8 and ϕ_2 , so that at $k=0$ it reduces to the energy current. The form of ψ_4 is

$$\psi_4(p) = \frac{(p/p_F)^3 \cos \theta [1 + b_4(x)]}{[N(0)N_4(x)]^{1/2}}. \quad (A4)$$

ψ_4 is orthogonal to ψ_1 and ψ_5 ; $b_4(x)$ is found by requiring ψ_4 to be orthogonal to ψ_2 . The result is

$$b_4(x) = x^2. \quad (A5)$$

The normalization constant $N_4(x)$ is given by

$$N_4(x) = \frac{1}{3}x^2(-2 + \frac{9}{5}x^2 + x^4). \quad (A6)$$

We write ψ_3 so that it reduces to $(p^2 - p_F^2)/2m^*$ at $k=0$:

$$\psi_3(p) = \frac{(p/p_F)^2 - [1 + b_3(x)] + (p/p_F)c_3(x)}{[N(0)N_3(x)]^{1/2}}. \quad (A7)$$

The functions b_3 and c_3 are chosen so that ψ_3 is orthogonal to ψ_1 and ψ_5 . The results are

*Work supported in part by NSF Grant No. DMR 76-21298, the Materials Research Laboratory of the NSF, and the Louis Block Fund of the University of Chicago.

¹R. Scherm, W. G. Stirling, A. D. B. Woods, R. A. Cowley, and G. J. Coombs, *J. Phys. C* **7**, 1341 (1974); W. G. Stirling, R. Scherm, P. A. Hilton, and R. A. Cowley, *J. Phys. C* **9**, 1643 (1976).

²K. Sköld, C. A. Pelizzari, R. Kleb, and C. E. Ostrowski, *Phys. Rev. Lett.* **37**, 842 (1976).

³See, for example, D. Pines and P. Nozières, *The Theory of Quantum Liquids* (Benjamin, New York, 1966), Vol. 1, p. 131.

⁴A. A. Abrikosov and I. M. Khakatnikov, *Rep. Prog. Phys.* **22**, 329 (1959).

⁵E. G. Flowers, R. W. Richardson, and S. J. Williamson,

Phys. Rev. Lett. **37**, 309 (1976).

⁶G. F. Mazenko, *Phys. Rev. A* **9**, 360 (1974); **7**, 209, 222 (1973).

⁷See, for example, A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Mechanics*, 2nd ed. (Pergamon, Oxford, 1965).

⁸G. Baym and L. P. Kadanoff, *Phys. Rev.* **124**, 287 (1961); G. Baym, *Phys. Rev.* **127**, 1391 (1962).

⁹G. F. Mazenko, *Phys. Rev. A* **3**, 2121 (1971).

¹⁰L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962).

¹¹G. F. Mazenko, Ref. 9 and Ph.D. thesis (MIT, 1970) (unpublished).

¹²C. D. Boley and J. B. Smith, *Phys. Rev. A* **12**, 661

- (1975).
- ¹³See Ref. 7, pp. 49–62, 152–176.
- ¹⁴E. P. Wigner, Phys. Rev. **40**, 749 (1932).
- ¹⁵We shall omit the arrows on vector quantities.
- ¹⁶For example, $L_I(k\bar{k}, p)f_{\bar{k}, -\bar{k}} \equiv \int d^3\bar{k} L_I(k\bar{k}, p)f_{\bar{k}, -\bar{k}}$.
- ¹⁷R. Hallock, J. Low Temp. Phys. **9**, 109 (1972).
- ¹⁸L. P. Kadanoff and P. C. Martin, Ann. Phys. (N.Y.) **24**, 419 (1963).
- ¹⁹See, for example, J. H. Irving and R. W. Zwanzig, J. Chem. Phys. **19**, 1173 (1951).
- ²⁰For example, $A(1\bar{1})B(1\bar{1}) = \int d^3r \int d^3p \bar{A}(1\bar{1})B(1\bar{1})$.
- ²¹Boley and Smith (Ref. 12) show that $\phi^{(s)}$ has a term of the form (4.2) in the weak-coupling approximation.
- ²²G. F. Mazenko, T. Y. C. Wei, and S. Yip, Phys. Rev. A **6**, 1981 (1972).
- ²³J. Sykes and G. A. Brooker, Ann. Phys. (N.Y.) **56**, 1 (1970).
- ²⁴The relaxation time τ is related to the ϕ_{11}, ϕ_{11} matrix element of $\phi^{(c)}$.
- ²⁵J. C. Wheatley, Rev. Mod. Phys. **47**, 415 (1975).
- ²⁶C. H. Aldrich, C. J. Pethick, and D. Pines, Phys. Rev. Lett. **37**, 845 (1976).
- ²⁷H. R. Glyde and F. C. Khanna, Phys. Rev. Lett. **37**, 1692 (1976).
- ²⁸A direct observation of the existence of zero sound was made by W. R. Abel, A. C. Anderson, and J. C. Wheatley, Phys. Rev. Lett. **17**, 74 (1966). The existence of zero sound at higher k was proposed by D. Pines, in *Quantum Fluids*, edited by D. F. Brewer (North-Holland, Amsterdam, 1965).
- ²⁹The fact that the theoretical $\omega_0(k)$ is larger than the experimental $\omega_0(k)$ implies that for fixed k the theoretical separation between the low ω peak in $S_I(k, \omega)$ and the zero-sound peak in $S(k, \omega)$ is larger than is observed experimentally. Thus, in our simple model, it is still possible to distinguish the two peaks in the total scattering function even at $T=0.63$ K. However, if we use as input the experimental values of $\omega_0(k)$, the two peaks merge.
- ³⁰See, for example, Ref. 3.
- ³¹K. Sköld and C. Pelizzari (unpublished).