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Dynamic correlations in Fermi fluids

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We consider dynamic correlations in normal liquid ^{3}He following the kinetic theory approach introduced in earlier work. The connection between the kinetic theory formalism and the Landau and polarization potential theories is discussed in detail and the approximations implicit in the latter approach are pointed out. We present a phenomenological approach to the dynamic memory function which incorporates finite frequency and wave-vector effects. This form of the dynamic kernel is used to evaluate the shear viscosity, including contributions from nonlocal effects. Estimates of these contributions indicate that they are non-negligible. The consequences of our method and its future application to the evaluation of collisional broadening of collective modes observed in neutron scattering experiments are discussed.

I. INTRODUCTION

Recent measurements[1,2] of inelastic neutron scattering from liquid ^{3}He have stimulated renewed interest in the theoretical description of the elementary excitations in normal Fermi liquids. Since a fully microscopic treatment of the static and dynamic properties of strongly interacting quantum liquids is difficult, most of the advances in understanding these properties have been from a semiphenomenological point of view. Landau theory[3] has provided a powerful description of the elementary excitations in the limit of small wave vector k and low frequency ω. However Landau Fermi-liquid theory is not applicable for the values of k and ω associated with the collective zero-sound mode excited by coherent neutron scattering. Pines and collaborators[4,5] have used a polarization potential approach to study intermediate wave-number phenomena not treated by the Landau theory. Using small and large wave-number data as input, this approach has been successfully used to analyze the observed zero-sound dispersion at intermediate k. The mechanism for the damping of zero sound in the polarization potential approach is the decay of zero sound into single particle-hole excitations. It appears likely, however, that in the range of wave-vector transfers of experimental interest, other dynamical damping mechanisms (collisions and multiparticle processes) must be taken into account.

In this paper we continue our quantum kinetic theory[6] approach to the dynamics of normal Fermi liquids and develop a more realistic treatment of quasiparticle collisions. In general the kinetic theory approach is characterized by a static "memory function" $\phi^{(s)}$ and a collisional memory function $\phi^{(c)}$. The static term describes instantaneous mean-field effects and reduces in the low temperature T and long-wavelength limit to the Landau quasiparticle interaction $f_{pp'}$. In our earlier work[6] (referred to in the following as I) we considered this correspondence in some detail and used it to formulate a phenomenological theory for $\phi^{(s)}$ applicable at intermediate wave vectors. The frequency-dependent term $\phi^{(c)}$ describes the effects of collisions and is a generalization of the frequency and wave number-independent Boltzmann-like collision integral considered in the Landau theory. The simple phenomenological model of $\phi^{(c)}$ adopted in I neglects the k and ω dependence of $\phi^{(c)}$ and assumes that the zero sound damping mechanism (in addition to Landau damping) is due to quasiparticle collisions within $k_B T$ of the Fermi surface. This thermal broadening of the quasiparticles, however, does not play a dominant role in the observed linewidth of zero sound at the wave vectors and temperatures of experimental interest.[1,2] Hence this simple model of $\phi^{(c)}$ is inadequate, and it is necessary to develop an approximation for $\phi^{(c)}$ which includes nonlocal frequency and wave-number effects.

In Sec. II we summarize the basic definitions characterizing the formal aspects of the kinetic theory. In Sec. III A we review our analysis of $\phi^{(s)}$ and discuss the limits in which our analysis reduces to the Landau theory. The analysis of the correspondence between the two k-dependent phenomenological parameters introduced in our model of $\phi^{(s)}$ and the first two sum rules for the dynamic structure function $S(k,\omega)$ is given in Appendix A. We formally compare the kinetic theory and polarization potential approaches in Sec. III B, show that the latter approach implicitly assumes that $\phi^{(c)}$ can be neglected, and determine the correspondence between the phenomenological parameters in the two theories. In Sec. IV we develop a phenomenological model of $\phi^{(c)}$ which includes binary quasiparticle collisions, incorporates nonlocal frequency and wave-number effects, and which reduces to the Landau and weak-coupling limits. As one test of this phenomenological form of $\phi^{(c)}$, we calculated in Sec. V the local an nonlocal contributions to the low-temperature shear viscosity and find a result consistent with experimental values; the nonlocal contribution is found to be nonnegligible. The details of the numerical calculation are given in Appendix B. Finally in Sec. VI we indicate how our approximate form for $\phi^{(c)}$ can be used to calculate the zero-sound dispersion and linewidth. A detailed calculation of the collisional broadening of the zero-sound mode and a comparison with neutron scattering results will be made in future work.

II. FORMAL DEVELOPMENT

A. Density fluctuations

The primary quantity of interest in this paper is the dynamic structure function $S(k,\omega)$ measured in coherent inelastic thermal neutron scattering.[1,2] Although $S(k,\omega)$ is a measure of the local density fluctuations, it is easier to construct a microscopic theory for a larger class of fluctuations. Consequently we consider fluctuations in the "phase-space density" or Wigner operator[7] defined at time t as

$$f(r,p,t)=\int\frac{d^3r'}{(2\pi)^3}e^{-i\vec{p}\cdot\vec{r}'}\psi^\dagger(r\ \tfrac{1}{2}r',t)\times\psi(r+\tfrac{1}{2}r',t)\ , \tag{2.1}$$

where the Heisenberg field operators $\psi(r,t)$ and $\psi^\dagger(r,t)$ satisfy the equal-time anticommutation relations for fermions. For simplicity we ignore spin until Sec. II B and set $\hbar$ and the volume of the system equal to unity unless otherwise noted. The usual number density operator is obtained by integrating $f(rp,t)$ over all p:

$$\psi^\dagger(\vec{r},t)\psi(\vec{r},t)=\int d^3p\, f(\vec{r},\vec{p},t)\ . \tag{2.2}$$

The dynamical fluctuations of a low-temperature quantum fluid are frequently investigated in terms of a generalized susceptibility χ defined as

$$\chi(11',t-t')=\Theta(t-t')\langle[f(1,t)f(1',t')]\rangle\ , \tag{2.3}$$

where $\langle\cdots\rangle$ denotes the thermal average in the grand canonical ensemble specified by the inverse temperature $\beta=(k_BT)^{-1}$ and chemical potential μ. In (2.3) $\Theta(t)$ is the usual step function, [ ] denotes the commutator, and $1\equiv(r_1,p_1)$, etc. The properties of χ are conveniently discussed in terms of its transforms:

$$\chi(k,pp',z)=\int d^3(r-r')\exp[-i\vec{k}\cdot(\vec{r}-\vec{r}')](-i)\int_0^\infty d(t-t')\exp[iz(t-t')]\chi(11',t-t')\ . \tag{2.4}$$

The relaxation or Kubo function[8] $\mathcal{L}$ is defined in terms of χ by

$$\mathcal{L}(k,pp',z)=(\beta z)^{-1}[\chi(k,pp',z)-\chi(k,pp',z=0)]\ . \tag{2.5}$$

The dynamic structure function $S(k,\omega)$ can be related to $\mathcal{L}$ using the fluctuation dissipation theorem[8] and is given by

$$nS(k,\omega)=\int d^3p\int d^3p'\,\beta\omega\coth\frac{\beta\omega}{2}(1+e^{-\beta\omega})^{-1}\,\mathrm{Im}\,\mathcal{L}(k,pp',z=\omega+i0^+)\ , \tag{2.6}$$

where n is the equilibrium averaged density.

The function $\mathcal{L}(k,pp',z)$ among the various two-point quantum correlation functions is the one which satisfies a well-behaved kinetic equation over a wide range of wave number, frquency, and temperature. The kinetic equation[6] for $\mathcal{L}(k,pp',z)$ has the general form[9]

$$\left[z-\frac{\vec{k}\cdot\vec{p}}{m}\right]\mathcal{L}(k,pp',z)-\phi(k,\bar{p}p',z)\mathcal{L}(k,\bar{p}p')=\tilde{\mathcal{L}}(k,pp')\ , \tag{2.7}$$

where

$$\tilde{\mathcal{L}}(k,pp')\equiv\mathcal{L}(k,pp',t=0)=-\beta^{-1}\chi(k,pp',z=0)\ . \tag{2.8}$$

The $z-\vec{k}\cdot\vec{p}/m$ term in (2.7) describes the motion of a single free-streaming particle of mass m. The second term in (2.7) represents the effects of collisions as $\mathcal{L}(k,pp',z)$ evolves from its initial condition $\tilde{\mathcal{L}}(k,pp')$. The memory function ϕ separates naturally into a sum of two terms,

$$\phi(k,pp',z)=\phi^{(s)}(k,pp')+\phi^{(c)}(k,pp',z)\ . \tag{2.9}$$

The frequency-independent or "static" term $\phi^{(s)}$

describes instantaneous mean-field effects and leads at long wavelengths to "dressed" particles, i.e., quasiparticles. An exact microscopic expression for $\phi^{(s)}$ is given by Eq. (3.16) of I, where it is also shown that this expression is equivalent to the condition

$$(\vec{k}\cdot\vec{p}/m)\chi(k,pp',z=0)+\phi^{(s)}(k,p\bar{p})\chi(k,\bar{p}p',z=0)=-\delta(p-p')[f(p-\tfrac{1}{2}k)-f(p+\tfrac{1}{2}k)] \quad . \tag{2.10}$$

The momentum distribution function $f(p)$ is the equilibrium average of the phase-space density operator

$$f(p)=\langle f(r,p,t)\rangle \quad . \tag{2.11}$$

Equation (2.10) can be interpreted as either an initial condition for $\mathcal{L}$ [see Eq. (2.8)] in terms of $\phi^{(s)}$ or as an integral equation for $\phi^{(s)}$. The latter interpretation of (2.10) is preferable for classical systems for which the momentum dependence of χ and f is known.

The frequency-dependent term $\phi^{(c)}$ describes the effects of collisions. The microscopic expression for $\phi^{(c)}(k,pp',z)$ can be written as[6]

$$\phi^{(c)}(1\bar{1},z)\tilde{\mathcal{L}}(\bar{1},1')=-L_I(1\bar{1})L_I(2\bar{2})\,\mathcal{G}(1\bar{1},2\bar{2},z) \quad , \tag{2.12}$$

where L_I is the interaction part of the two-particle Liouville operator and $\mathcal{G}$ represents the dynamical evolution of two particles. The microscopic defintions of L_I and $\mathcal{G}$ are given in I. Equation (2.12) is the basis of our phenomenological theory for $\phi^{(c)}$ presented in Sec. IV.

B. Spin-density fluctuations

We briefly discuss the extension of the above formalism to spin $\frac{1}{2}$. The spin-dependent Wigner operators are defined as

$$F_\alpha(1,t)=\int\frac{d^3r'}{(2\pi)^3}e^{-i\vec{p}\cdot\vec{r}'}\psi^\dagger_\alpha(r-\tfrac{1}{2}r',t)\times\psi_\alpha(r+\tfrac{1}{2}r',t) \quad , \tag{2.13}$$

where α is a spin index which is represented as $\uparrow$ or $\downarrow$. The definitions of the correlation functions can be generalized in an obvious way, e.g.,

$$\chi_{\alpha\beta}(11',t-t')=\Theta(t-t')\langle[f_\alpha(1,t),f_\beta(1',t')]\rangle \quad . \tag{2.14}$$

The dynamic structure function $S(k,\omega)$ becomes the symmetric combination

$$S(k,\omega)=\tfrac{1}{2}[S_{\uparrow\uparrow}(k,\omega)+S_{\uparrow\downarrow}(k,\omega)] \quad . \tag{2.15}$$

The antisymmetric combination

$$S_\sigma(k,\omega)=\tfrac{1}{2}[S_{\uparrow\uparrow}(k,\omega)-S_{\uparrow\downarrow}(k,\omega)] \tag{2.16}$$

is associated with spin-density fluctuations. The total differential cross section for the scattering of unpolarized neutrons from liquid ^{3}He is

$$S_T(k,\omega)=S(k,\omega)+(\sigma_i/\sigma_c)S_\sigma(k,\omega) \quad , \tag{2.17}$$

where σ_i and σ_c are the incoherent and coherent cross sections, respectively; the latter term in (2.17) arises from incoherent scattering.

The formalism of Sec. II A applies of $\mathcal{L}$ and $\mathcal{L}_\sigma$ separately. For example $\mathcal{L}_\sigma$ satisfies an equation of the form (2.7) with memory function ϕ_σ. Since the interparticle potential in liquid ^{3}He is assumed to be spin independent, the remaining complications of spin can be taken into account by associating a factor of 2 with each momentum integral. However, we shall not explicitly include these factors in the following unless otherwise noted.

III. PHENOMENOLOGICAL MEAN-FIELD THEORIES

The problem of calculating the dynamic structure function $S(k,\omega)$ has been shifted to calculating the quantities f, $\phi^{(s)}$ and $\phi^{(c)}$ which enter into the kinetic equation (2.7) determining the Kubo response function. We consider first in this section the correspondence of our formal development to Landau's phenomenological theory of a normal Fermi liquid. This correspondence is used in Sec. III A to develop a phenomenological theory of f and $\phi^{(s)}$ which is applicable to liquid ^{3}He at finite wave vectors. Our generalization of the Landau theory is compared to the polarization potential approach[4] in Sec. III B. The development of a phenomenological model for $\phi^{(c)}$ is given in Sec. IV.

A. Static memory function and the Landau theory

In the Landau limit of low temperature, small wave vector and low frequency, normal liquid ^{3}He can be represented as a dilute gas of quasiparticles with an effective mass m^* different from the bare mass m of an individual ^{3}He atom. Hence we assume that the equilibrium one-particle distribution function $f(p)$ defined by (2.11) can be represented by

$$f(p)=(2\pi)^{-3}\{\exp[\beta(p^2/2m^*-\mu)]+1\}^{-1} \quad . \tag{3.1}$$

As inspection of (2.6) shows that for the particles to acquire a mass m^*, $\phi^{(s)}$ must take the general form

$$\phi^{(s)}=\frac{\vec{k}\cdot\vec{p}}{m'}(p-p')+\frac{\vec{k}\cdot\vec{p}}{m^*}\phi^*(k,pp') \quad , \tag{3.2}$$

with m^* independent of k and $m'^{-1}=m^{*-1}-m^{-1}$. The definition of ϕ^* in (3.2) allows us to rewrite (2.10) as

$$\chi(k,pp',0)+\phi^*(k,p\bar{p})\chi(k,\bar{p}p',0) = -F_k(p)\delta(p-p') \ , \quad (3.3)$$

where

$$F_k(p)=\frac{m^*}{\vec{k}\cdot\vec{p}}\left[f\left(p-\frac{k}{2}\right)-f\left(p+\frac{k}{2}\right)\right] \ , \quad (3.4)$$

and $f(p)$ is given by (3.1).

There is a close connection between the $k\rightarrow 0$ limit of $\phi^*(k,pp')$ and the Landau quasiparticle interaction energy[3] $f_{pp'}$. The connection can be made directly by substituting (3.2) in (2.7) and ignoring $\phi^{(c)}$. In this way we obtain the usual (collisionless) Landau kinetic equation with $\phi^*(0,pp')$ identified with $f_{pp'}$. We know that for an isotropic system and with p and p' on the Fermi surface, $f_{pp'}$ depends only on the angle between p and p' and can be expanded in Legendre polynomials P_l as

$$f_{pp'}=[\nu(0)]^{-1}\sum_l F_l P_l(\hat{p}\cdot\hat{p}') \ . \quad (3.5)$$

The F_l are the usual (dimensionless) Landau parameters; $\nu(0)$ is the density of quasiparticle states at the Fermi surface:

$$\nu(0)=m^* p_F/\pi^2 \ , \quad (3.6)$$

where p_F is the Fermi momentum.

For $k\neq 0$ we expand $\phi^*(k,pp')$ in a manner analogous to (3.5). That is, we represent ϕ^* in terms of its matrix elements which reduce to the Landau parameters in the $k,T\rightarrow 0$ limit. Let us introduce a linear vector space $|p\rangle$ satisfying

$$\langle p|p'\rangle=\delta(p-p')F_k(p) \ , \quad (3.7)$$

$$|\bar{p}\rangle[F_k(\bar{p})]^{-1}\langle\bar{p}|=1 \ . \quad (3.8)$$

For any function $f(k,pp',z)$ we define an operator $f_k(z)$ such that

$$\langle p|f_k(z)|p'\rangle=f(k,pp',z)F_k(p) \ . \quad (3.9)$$

The momentum-dependent basis functions $\psi_i(k,p)$ are defined as projections onto $|p\rangle$:

$$F_k(p)\psi_i(k,p)\equiv\langle i|p\rangle \ . \quad (3.10)$$

We choose the set $\{\psi_i\}$ to be a complete set, normalized with respect to $F_k(p)$:

$$\psi_i(k,\bar{p})\psi_j(k,\bar{p})F_k(\bar{p})=\delta_{i,j} \ , \quad (3.11)$$

$$\sum_{i=1}^{\infty}\psi_i(k,p)\psi_i(k,p')F_k(p)=\delta(p-p') \ . \quad (3.12)$$

The set $\{\psi_i\}$ is ordered such that the first five functions ($i=0,\ldots,4$) are proportional to the hydrodynamical functions corresponding to conservation of particles, momentum, and energy. The explicit forms of ψ_0 and ψ_1 at $T=0$ are

$$\psi_0(k)=[\nu(0)L(x)]^{-1/2} \ , \quad (3.13)$$

$$\psi_1(k,p)=[3/\nu(0)]^{1/2}p\cos\theta/p_F \ , \quad (3.14)$$

where

$$L(x)=\begin{cases}\frac{1}{2}\left[1+\frac{1-x^2}{2x}\ln\left(\frac{1+x}{1-x}\right)\right], & x\leq 1\\ \frac{1}{2}, & x\geq 1\end{cases} \ , \quad (3.15)$$

with $x=k/2p_F$, and the z axis is along k. Note that ψ_0 is independent of p.

The above formal definitions allow us to represent[10] $\phi^*(k,pp')$ as

$$\phi^*(k,pp')=\sum_{ij}\psi_i(k,p)\psi_j(k,p')\,\phi^*_{ij}(k)F_k(p) \ , \quad (3.16)$$

where the matrix elements ϕ^*_{ij} are given by

$$\phi^*_{ij}(k)=\int d^3p\,d^3p'\,\psi_i(k,p)\psi_j(k,p')\phi^*(k,pp')F_k(p') \ . \quad (3.17)$$

In analogy to the Landau theory in which the expansion (3.5) of f'_{pp} is truncated after the first two terms, we consider the approximation in which only the first two diagonal matrix elements in (3.16) are retained. We first consider the spin-symmetric part of ϕ^* and write

$$\phi^*(k,pp')=[\alpha_0(k)\psi_0^2+\alpha_1(k)\psi_1(p)\psi_1(p')]F_k(p) \ . \quad (3.18)$$

It is easy to see that $\alpha_0(k=0)=F_0^s$ and $\alpha_1(k=0)=F_1^s/3$. This limiting behavior can be determined by recalling the correspondence between $\phi^*(0,pp')$ and $f_{pp'}$ and noting that the angular dependence of ψ_0 and ψ_1 corresponds to the $l=0$ and $l=1$ Legendre polynomials. We could obtain the k dependence of α_0 and α_1 through a direct experimental determination of $\chi(k,pp',0)$ whose functional dependence on $\alpha_0(k)$, $\alpha_1(k)$ is determined by (3.3) and (3.18). Since such data are not available, we rely on the sum rules

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S(k,\omega)=S(k) \ , \quad (3.19)$$

and

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega\tanh\frac{\beta\omega}{2}S(k,\omega)=\frac{k^2}{2m} \quad (3.20)$$

to indirectly determine $\alpha_0(k)$ and $\alpha_1(k)$. We solve for $S(k,\omega)$ in terms of α_0 and α_1 using (2.6) and (2.7) and the assumptions that ϕ^* is given by (3.18) and that $\phi^{(c)}$ can be neglected in this context. The details of this solution are given in Appendix A. We find that at $T=0$ within these approximations

$$S(k) = -\frac{1}{\pi}\nu(0)L(x) \times \int_0^\infty dy \frac{I_0^2(y)(1+\alpha_0+\alpha_0\alpha_1+\alpha_1 a^2y^2)}{[1+\alpha_0 I_0(y)+(1+\alpha_0)\alpha_1 I_1(y)]} , \quad (3.21)$$

and

$$\frac{k^2}{2m^*}[1+\alpha_1(k)] = \frac{k^2}{2m} . \quad (3.22)$$

The k- and y-dependent functions I_0 and I_1 are defined in Appendix A; $a^2 = 3m^*/k^2p_F^2$. For $k \to 0$, we can write (3.21) as

$$S(k) = \alpha(3k/4p_F) , \quad (3.23)$$

where the constant α agrees with the prediction[11] of the Landau theory. The relation (3.22) together with the Landau result $m^*/m = (1+F_1^s/3)$ implies that within the present approximation

$$\alpha_1(k) = F_1^s/3 \quad (3.24)$$

for k in the range considered. We use the measured values[12] of $S(k)$ at $T \sim 0.4$ K in the range $0.15 \le k \le 2.1$ Å^{-1} and the value $m^* = 3.08m$ as input to determine the k dependence of $\alpha_0(k)$ from (3.21). A plot of $\alpha_0(k)/L(k/2p_F)$ is shown as the solid line in Fig. 1. (The function α_0/L rather than α_0 itself is shown in Fig. 1 in order to make a later comparison to a corresponding phenomenological parameter in the theory[4] of Aldrich and Pines.) It is seen from Fig. 1 that α_0 has a maximum for $k \sim 0.8$ Å^{-1} and goes to zero as $S(k)$ approaches unity for large k. The extrapolated value of $\alpha_0(k)$ to $k=0$ agrees with $F_0^s = 10.07$ from the Landau theory. The parameter $\alpha_0(k)$ can be interpreted as the finite-k generalization of F_0^s.

The basic assumption in our determination of α_0 and α_1 is the neglect of $\phi^{(c)}$, an assumption consistent with Landau theory. We know from I that the determination of α_0 and α_1 is unchanged if $\phi^{(c)}$ is included in a frequency-independent approximation [see (4.1) in Sec. IV]. However, in general the inclusion of a frequency-dependent $\phi^{(c)}$ would modify the sum-rule determination of α_0 and α_1.

The convergence of representations such as (3.16) is well known in classical kinetic theory but is unknown in the present context of the determination of ϕ^*. Our main assumption was to ignore all matrix elements in (3.16) for $i,j > 1$. It would be possible to retain additional matrix elements, e.g., ϕ_{22}^* and ϕ_{33}^* corresponding to the transverse momenta p_x and p_y and ϕ_{44}^* corresponding to the energy, and to determine their magnitude from sum rules for the

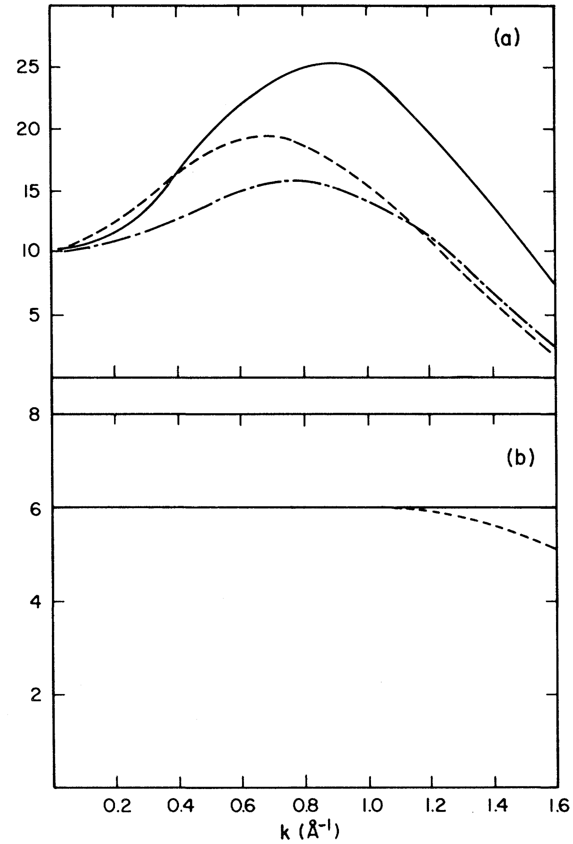

FIG. 1. Top: The parameter $\alpha_0(k)$ divided by $L(k/2p_F)$ (solid line) as a function of k. The other two lines represent [see Eq. (3.49)] the corresponding quantity from polarization potential theory: the dashed line is from Ref. 4, the dash dotted line from Ref. 14. Bottom: the same comparison for $\alpha_1/(1+\alpha_1)$.

transverse-current correlation function and the sum rule for $S(k,\omega)$. However since a two-parameter theory is known to give reasonable results in the Landau limit, we postpone an investigation of the higher-order matrix elements to future work.

We model the spin-antisymmetric part of ϕ^* in an analogous manner to the symmetric part and write

$$\phi_\sigma^*(k,pp') = [\beta_0(k)\psi_0^2 + \beta_1(k)\psi_1(p)\psi_1(p')]F_k(p) , \quad (3.25)$$

where we have denoted the first two diagonal matrix elements as $\beta_0(k)$ and $\beta_1(k)$, respectively. The parameters β_0 and β_1 reduce at $k=0$ to the spin antisymmetric Landau parameters F_0^a and $F_1^a/3$, respectively. For $k \ne 0$ the determination of β_0 and β_1 is limited by the fact that the k dependence of the incoherent or spin-symmetric static-structure function $S_\sigma(k)$ is not accurately known at present. In addition no f-sum rule analogous to (3.20) can be used, since the spin current is not a conserved quantity. Hence the procedure that we used to determine α_0 and α_1 implies only that $\beta_0(k) \to 0$ as $S_\sigma(k) \to 1$ for large enough k. The results[4] of Aldrich and Pines and the investigations of other models[13] indicate that β_0 and β_1 are slowly varying functions of k for $k \leq 2p_F$. Since we have no other *a priori* information on the k dependence of β_0 and β_1, we assume that

$$\beta_0(k) = F_0^a \ , \qquad \beta_1(k) = F_1^a/3 \ , \tag{3.26}$$

for $k \lesssim 2p_F$. We find *a posteriori* in Sec. V this assumption is consistent with experimental results for the viscosity.

B. Comparison to the polarization potential approach

We now formally compare our kinetic theory approach to the polarization potential[4] method. In the latter method the effects of the interaction are described in terms of k-dependent self-consistent fields. The basic assumption is that $\chi(k,z)$ [see Eq. (3.31)] can be expressed as

$$\chi(k,z) = \frac{\chi_{\rm sc}(k,z)}{1 - [f_k^s + (z^2/k^2)g_k^s]\chi_{\rm sc}(k,z)} \ , \tag{3.27}$$

where $\chi_{\rm sc}$ is a screened response function. The phenomenological polarization parameters, f_k^s and g_k^s, are determined in part by analogy to the corresponding effective interaction in liquid ^{4}He.

In order to identify $\chi_{\rm sc}$, f_k^s, and g_k^s with the microscopic quantities appearing in the kinetic theory development, we formally solve (2.5), (2.7), and (3.3) for $\chi(k,pp',z)$. The reader not interested in this formal derivation may skip to (3.47). We use the formal vector space introduced in (3.7)–(3.9) and the definition (3.2) of ϕ^* to write (2.7) and (3.3) in operator notation as

$$[z - \omega_k(1+\phi_k^*) - \phi_k^{(c)}(z)]\mathfrak{L}_k(z) = \tilde{\mathfrak{L}}_k \ , \tag{3.28a}$$

$$(1+\phi_k^*)\chi_k(z=0) = -F_k \ , \tag{3.28b}$$

or using the relation (2.9)

$$L_k(z) = [z - \omega_k(1+\phi_k^*) - \phi_k^{(c)}(z)]^{-1}\beta^{-1}(1+\phi_k^*)^{-1}F_k \tag{3.29}$$

In the above $\langle p|\omega_k|p'\rangle = (\vec{k}\cdot\vec{p}/m^*)\delta(p-p')F_k(p)$. From (2.5) and (3.29) we have

$$\chi_k(z) = [z - \omega_k(1+\phi_k^*) - \phi_k(z)]^{-1} \times [\omega_k + \phi_k^{(c)}(z)(1+\phi_k^*)^{-1}]F_k \ . \tag{3.30}$$

The quantity of interest $\chi(k,z)$ is formally related to the operator $\chi_k(z)$ by

$$\chi(k,z) = \int d^3p\, d^3p' \langle p|\chi_k(z)|p'\rangle F_k^{-1}(p') \ , \tag{3.31}$$

so that using (3.10) and (3.30) we obtain

$$\chi(k,z) = \nu(0)L(x)\langle 0|[z - \omega_k(1+\phi_k^*) - \phi_k^{(c)}(z)]^{-1}[\omega_k + \phi_k^{(c)}(z)(1+\phi_k^*)^{-1}]|0\rangle \ . \tag{3.32}$$

The condition for conservation of particles,

$$\int d^3p'\, d^3\bar{p}\, \phi^{(c)}(k,p\bar{p},z)\tilde{\mathfrak{L}}(k,\bar{p}p') = 0 \ ,$$

implies that the term $\phi_k^{(c)}(z)(1+\phi_k^*)^{-1}|0\rangle$ in (3.32) can be omitted. Thus we have the simple but formal result

$$\chi(k,z) = \nu(0)L(x)\langle 0|[z - \omega_k(1+\phi_k^*) - \phi_k^{(c)}(z)]^{-1}\omega_k|0\rangle \ . \tag{3.33}$$

Although (3.33) is completely general, we need to analyze it further in order to write it in the form (3.27). We write

$$\omega_k\phi_k^* + \phi_k^{(c)}(z) = \omega_k\phi_k^0 + \Sigma_k(z) \ , \tag{3.34}$$

where ϕ_k^0 is the operator analog of the approximation (3.18) to ϕ_k^*; the remainder of $\omega_k\phi_k^*$ added to $\phi_k^{(c)}(z)$ constitutes $\Sigma_k(z)$. It is convenient to define a generalization of (3.33), e.g., the matrices

$$G_{ij}(k,z) = \langle i|[z - \omega_k(1+\phi_k^0) - \Sigma_k(z)]^{-1}\omega_k|j\rangle \ , \tag{3.35a}$$

and

$$G_{ij}^{(0)}(k,z) = \langle i|[z - \omega_k - \Sigma_k(z)]^{-1}\omega_k|j\rangle \ , \tag{3.35b}$$

then

$$\chi(k,z) = \nu(0)L(x)G_{00}(k,z) \ . \tag{3.36}$$

G_{ij} and $G_{ij}^{(0)}$ are related by the system of equations

$$G_{ij} = G_{i,j}^{(0)} + \alpha_0 G_{i,0}^{(0)}G_{0,j} + \alpha_1 G_{i,1}^{(0)}G_{1,j} \ , \tag{3.37}$$

which can be solved for G_{00}:

$$G_{00}(k,z) = \frac{G_{00}^{(0)}(k,z) - \alpha_1(k)N(k,z)}{1 - \alpha_0(k)G_{00}^{(0)}(k,z) - \alpha_1(k)G_{11}^{(0)}(k,z) + \alpha_0(k)\alpha_1(k)N(k,z)} \ . \tag{3.38}$$

In the above $N(k,z)$ can be expressed as

$$N(k,z)=G_{00}^{(0)}G_{11}^{(0)}-G_{10}^{(0)}G_{01}^{(0)} \ . \tag{3.39}$$

We now show that we can express $G_{11}^{(0)}$, $G_{10}^{(0)}$, and $G_{01}^{(0)}$ in terms of $G_{00}^{(0)}$ and hence simplify (3.39). If we use the identity $z/(z-\Delta)=1+\Delta/(z-\Delta)$, we can rewrite $G_{00}^{(0)}$ defined in (3.35b) as

$$zG_{00}^{(0)}(k,z)=\langle 0|\omega_k|0\rangle+\langle 0|\omega_k+\Sigma_k(z)[z-\omega_k-\Sigma_k(z)]^{-1}\omega_k|0\rangle \ . \tag{3.40}$$

Using the fact that $\langle 0|\Sigma_k(z)=0$, which follows from conservation of particles, and the property $\langle 0|\omega_k|0\rangle=0$ we can write (3.40) as

$$zG_{00}^{(0)}(k,z)=\langle 0|\omega_k[z-\omega_k-\Sigma_k(z)]^{-1}\omega_k|0\rangle \ . \tag{3.41}$$

Since $\langle p|\omega_k|0\rangle=\kappa\langle p|1\rangle$, where

$$\kappa=kp_F/\{m^*[3L(x)]^{1/2}\} \tag{3.42}$$

we obtain the relation

$$G_{10}^{(0)}(k,z)=\frac{z}{\kappa}G_{00}^{(0)}(k,z) \ . \tag{3.43}$$

In the same way it can also be shown that

$$G_{11}^{(0)}(k,z)=\frac{z^2}{\kappa^2}G_{00}^{(0)}(k,z)-\left\{1+\frac{1}{\kappa}R(k,z)\right\} \ , \tag{3.44}$$

where

$$R(k,z)=\frac{1}{\kappa}\langle|[z-\omega_k-\Sigma_k(z)]^{-1}\Sigma_k(z)\omega_k|0\rangle \ . \tag{3.45}$$

We substitute (3.43), (3.44), and a similar identity for $G_{01}^{(0)}$ into (3.39) and obtain after some algebra that

$$N(k,z)=-G_{00}^{(0)}(k,z) \ . \tag{3.46}$$

The desired form for $\chi(k,z)$ is obtained from (3.36), (3.38), (3.44), and (3.46) and can be expressed as

$$\chi(k,z)=\frac{\chi_{sc}(k,z)}{\left[1-\left\{\alpha_0+\frac{\alpha_1}{1+\alpha_1}\frac{z^2}{\kappa^2}\right\}\chi_{sc}(k,z)\nu(0)^{-1}L_{(x)}^{-1}+\frac{\alpha_1}{1+\alpha_1}R(k,z)\right]} \ , \tag{3.47}$$

where

$$\chi_{sc}(k,z)=\nu(0)L(x)G_{00}^{(0)}(k,z) \ . \tag{3.48}$$

A comparison of the formal exact result (3.47) with the polarization potential form (3.27) shows that the latter requires that $R(k,z)=0$. From (3.45) it follows that $\Sigma_k(z)$, i.e., $\phi_k^{(c)}(z)$ plus the corrections to (3.18), must be set equal to 0. Hence as expected (3.27) represents a generalized mean field or random phase approximation. The parameters f_k^s and g_k^s in (3.27) are related to $\alpha_0(k)$ and $\alpha_1(k)$ by

$$\alpha_0(k)\rightarrow\nu(0)L(x)f_k^s \ , \tag{3.49}$$

$$\frac{\alpha_1(k)}{1+\alpha_1(k)}\rightarrow\frac{n}{m}g_k^s\frac{m}{m^*} \ . \tag{3.50}$$

Note that if one sets $m^*/m=1+\alpha_1(k)$ in (3.50), then $\alpha_1(k)=(n/m)g_k^s$. However, this relation between m^* and α_1 is not exact at larger k.

It is interesting to compare using (3.49) and (3.50) the k dependence of α_0 and α_1 determined approximately in Sec. III A with the k dependence of the polarization potential parameters f_k^s and g_k^s determined independently by Pines and collaborators.[4] In Fig. 1(a) we plot the quantity $\alpha_0(k)/L(x)$ (solid line) and compare it with f_k^s. The dashed curve is taken from Aldrich and Pines[4] and the dash-dotted line from Bedell and Pines.[14] There is qualitative agreement among the three plots. The form of f_k^s assumed by Aldrich and Pines yields good agreement with experiment for the dispersion relation of the zero-sound mode, in contrast to the behavior of α_0 which leads to a dispersion relation that is too high for $k\sim p_F$. The values of f_k^s used by Bedell and Pines to calculate the transport coefficients have not been used[15] to calculate the zero-sound dispersion relation. It would appear that unless other polarization potential parameters such as the "multipair contributions" to χ_{sc} are also modified, the agreement obtained by Aldrich and Pines might be lost. In Fig. 1(b) we compare $\alpha_1/(1+\alpha_1)$ with ng_k^s/m^* and find qualitative agreement. As discussed in Sec. III A, knowledge of the high-frequency behavior of $\phi^{(c)}$ would likely modify the large-k behavior of α_0 and α_1. The question of whether development of a more realistic frequency-dependent form for $\phi^{(c)}$ and further refinements in the polarization potential theory will lead to closer agreement of the two methods and to closer agreement with experiment will be investigated in future work.

The spin antisymmetric parameters β_0 and β_1 can be related to f_k^a and g_k^a in an analogous manner. In both theories the determination of the spin antisymmetric parameters is much more difficult and uncertain. The form of f_k^a which yields the best values for the transport coefficients[14] is a slowly varying function of k in the range $0 < k \lesssim 2p_F$; g_k^a is taken to be identically zero[4] or a small quantity.[14] These assumptions are consistent with our assumption (3.26). As we discuss in Sec. V, the k dependence of β_0 for $k > 2p_F$ does affect the transport coefficients.

IV. COLLISIONAL PART OF THE MEMORY FUNCTION

In our earlier work[6] the collisional part of the memory function was represented by the simple form

$$\phi^{(c)}(k,pp',z) = -i\tau_\eta^{-1}\left[\delta(p-p') - \sum_{i=0}^{4}\psi_i(p)\psi_j(p')F_k(p)\right] , \tag{4.1}$$

where the relaxation time τ_η is independent of k and z and associated with the shear viscosity. The form (4.1) is consistent with the conservation laws and τ_η^{-1} is proportional to T^2 for low T. However, as discussed in Sec. I, the width of the zero-sound resonance as observed in the neutron scattering experiments is dominated by the ω dependence of $\phi^{(c)}$ rather than by its T dependence. Consequently we wish to develop a phenomenological theory for $\phi^{(c)}$ which includes finite k and ω effects.

Let us first investigate the form of $\phi^{(c)}$ in the weak-coupling limit. From (2.12) we know that the combination

$$K(11',z) = \phi^{(c)}(1\bar{1},z)\tilde{L}(\bar{1}1') , \tag{4.2}$$

rather than $\phi^{(c)}$ itself enters into the analysis. To include the effects of spin we write (for the symmetric part of K)

$$K(11',z) = \tfrac{1}{2}[K_{\uparrow\uparrow}(11',z) + K_{\uparrow\downarrow}(11',z)] . \tag{4.3}$$

Since the weak coupling limit of K is similar to that of Ref. 16, we only present the results here. We write

$$K(k,pp',z) = -\frac{\beta^{-1}}{2}\int d^3p_1\,d^3p_2\,d^3p_3\,d^3p_4\,[\tfrac{1}{2}W_{\uparrow\uparrow}(1234p)^2 + W_{\uparrow\downarrow}(1234p)^2][G(1234;Kz) - G(1234,-k-z)] , \tag{4.4}$$

where

$$\begin{aligned} W_{\uparrow\uparrow}(1234p) = {} & [V(p_1-p_3) - V(p_1-p_4)]\delta(p-p_1) + [V(p_2-p_4) - V(p_2-p_3)]\delta(p-p_2) \\ & - [V(p_3-p_1) - V(p_3-p_2)]\delta(p-p_3) - [V(p_4-p_2) - V(p_4-p_1)]\delta(p-p_4) , \end{aligned} \tag{4.5}$$

$$G(1234;kz) = (2\pi)^3\frac{\delta(p_1+p_2-p_3-p_4+k)B(1234,k)}{[z-E(1234,k)]E(1234,k)} , \tag{4.6}$$

$$B(1234,k) = \frac{1}{2}\left[f\left(p_1+\frac{k}{2}\right)f\left(p_2+\frac{k}{2}\right)\tilde{f}\left(p_3-\frac{k}{2}\right)\tilde{f}\left(p_4-\frac{k}{2}\right) - \tilde{f}\left(p_1+\frac{k}{2}\right)\tilde{f}\left(p_2+\frac{k}{2}\right)f\left(p_3-\frac{k}{2}\right)f\left(p_4-\frac{k}{2}\right)\right] , \tag{4.7}$$

and

$$E(1234,k) = \frac{1}{2m}\left[\left(p_1+\frac{k}{2}\right)^2 + \left(p_2+\frac{k}{2}\right)^2 - \left(p_3-\frac{k}{2}\right)^2 - \left(p_4-\frac{k}{2}\right)^2\right] . \tag{4.8}$$

We have used the notation

$$\tilde{f}(p) = 1-(2\pi)^3 f(p) , \tag{4.9}$$

where $f(p)$ is given by (3.1) with m^* replaced by m, and $V(k)$ is the Fourier transform of the interparticle potential $V(r)$. $W_{\uparrow\downarrow}$ has the same form as (4.5) except for the absence of the exchange terms. Note that in the limit $k,z \to 0$, K reduces to the familiar Boltzmann collision kernel with the transition probability evaluated in the Born approximation.

The form (4.4) of K is not applicable to liquid ^{3}He which is a strongly coupled Fermi liquid. The next step in a complete theory is to develop a systematic approximate procedure for $\phi^{(c)}$ applicable to realistic potentials. Although we have made some progress in this direction, we proceed instead to use simple arguments based on the weak coupling limit and the Landau theory to develop a phenomenological model of $\phi^{(c)}$ which is applicable to low-temperature liquid ^{3}He and which retains nonlocal effects. We know that the Landau collision integral[3] is of the Boltzmann form with a screened quasiparticle interaction, e.g., the "bare" interaction F_0^s is replaced by $F_0^s/(1+F_0^s)$. In order to retain the simplicity of the Landau collision integral, we assume a weak coupling form for $\phi^{(c)}$ in which the bare potential is replaced by an effective interaction. Hence we assume that the form of K is given by (4.4) with m replaced by m^* in (4.8) for E and $f(p)$ given by (3.1). A form of the effective interactions $W_{\uparrow\uparrow}$ and $W_{\uparrow\downarrow}$ that satisfies the symmetry conditions

$$W(1234p) = -W(3412p) = -W(2134p) \quad , \tag{4.10}$$

and the requirement derived from number conservation

$$\int d^2p\, W(1234p) = 0 \quad , \tag{4.11}$$

is given by

$$\begin{aligned} W(1234p) = \frac{1}{2}\Bigg\{ & \left[A\left(p_3-p_1;p_1-\frac{k}{2},p_2+\frac{k}{2}\right) + A\left(p_3-p_1;p_3+\frac{k}{2},p_4-\frac{k}{2}\right)\right]\delta(p-p_1) \\ & + \left[A\left(p_4-p_2;p_1+\frac{k}{2},p_2-\frac{k}{2}\right) + A\left(p_4-p_2;p_3-\frac{k}{2},p_4+\frac{k}{2}\right)\right]\delta(p-p_2) \\ & - \left[A\left(p_1-p_3;p_3+\frac{k}{2},p_4-\frac{k}{2}\right) + A\left(p_1-p_3;p_1-\frac{k}{2},p_2+\frac{k}{2}\right)\right]\delta(p-p_3) \\ & - \left[A\left(p_2-p_4;p_3-\frac{k}{2},p_4+\frac{k}{2}\right) + A\left(p_2-p_4;p_1+\frac{k}{2},p_2-\frac{k}{2}\right)\right]\delta(p-p_4)\Bigg\} \quad . \end{aligned} \tag{4.12}$$

The "scattering amplitude" $A(q;p,p')$ in (4.12) depends on the momentum transfer q and the momenta p and p' of the incoming particles. The first term in (4.12) corresponds to the process in which particles of momenta $p_1-k/2$ and $p_2+k/2$ scatter into final states $p_3-k/2$ and $p_2+k/2$ with momentum transfer p_3-p_1. In the weak coupling limit $A_{\uparrow\uparrow}(q,pp') = V(q) - V(p'-p-q)$ and the form (4.12) for W reduces to (4.5); the weak coupling limit of $A_{\uparrow\downarrow}(q,pp') = V(q)$. In general A also depends on the energy transfer. These dynamical effects are neglected here since they do not contribute to the low-temperature transport coefficients.[3]

In order to determine the form of the effective interaction $A(q,pp')$ between the quasiparticles, we use an analogy[3] between the form of A in the Landau theory and the screened interaction in the classical electron gas. The simplest form[17] of the static effective interaction $V_{\rm eff}$ for the latter system is

$$V_{\rm eff}(k) = -\beta^{-1}C(k)/\epsilon(k) \quad , \tag{4.13}$$

where the static dielectric function $\epsilon(k)$ is given by

$$\epsilon(k) = 1 - nC(k) \quad , \tag{4.14}$$

and $C(k)$ is the direct correlation function. The static part of the classical memory function can also be exactly expressed[18] in terms of $C(k)$

$$\phi^{(s)}(k,pp') = -\frac{\vec{k}\cdot\vec{p}}{m}C(k)\,n\,\phi_0(p) \quad , \tag{4.15}$$

where $\phi_0(p)$ is the Maxwellian. We ignore spin for the moment and generalize the relations (4.13)–(4.15) so that A, C, and ϵ depend on p and p' as well as q but retain essentially the same form. The generalized direct correlation function $C(k,pp')$ is determined by comparing (4.15) with our approximate form (3.18) for the quasiparticle static memory function. We take

$$\begin{aligned} C(k,pp') = -[&\alpha_0\psi_0(p)\psi_0(p') \\ &+ \alpha_1\psi_1(p)\psi_1(p')]F_k(p) \quad , \end{aligned} \tag{4.16}$$

and use the analogies based on (4.14) and (4.15) to write

$$A(k,pp')F_k(p') = -C(k,p\bar{p})E(k,\bar{p}p')^{-1} \tag{4.17}$$

with

$$E(k,pp') = \delta(p-p') - C(k,pp') \quad . \tag{4.18}$$

In order to solve (4.17) for $A(k,pp')$, we evaluate $E(k,pp')^{-1} = I(k,pp')$ from (4.16) and (4.18). $I(k,pp')$ satisfies the equation

$$[\delta(p-\bar{p}) + \alpha_0\psi_0(p)\psi_0(\bar{p})F_k(p) + \alpha_1\psi_1(p)\psi_1(\bar{p})F_k(p)]\,I(k,\bar{p}p') = \delta(p-p') \quad , \tag{4.19}$$

which can be easily solved to find

$$I(k,pp') = E(k,pp')^{-1} = \delta(p-p') - \frac{\alpha_0}{1+\alpha_0}\psi_0(p)\psi_0(p')F_k(p) - \frac{\alpha_1}{1+\alpha_1}\psi_1(p)\psi_1(p')F_k(p) \quad . \tag{4.20}$$

We solve (4.17) for $A(k,pp')$ using (4.16) and (4.20) and obtain

$$A(k,pp') = \frac{\alpha_0}{1+\alpha_0}\psi_0(p)\psi_0(p') + \frac{\alpha_1}{1+\alpha_1}\psi_1(p)\psi_1(p') \ . \tag{4.21}$$

The spin can now be included using arguments similar to those in the Landau theory. We write

$$A_{\sigma\sigma'}(k,pp') = \left[\frac{\alpha_0(k)}{1+\alpha_0(k)} + \sigma\sigma'\frac{\beta_0(k)}{1+\beta_0(k)}\right]\psi_0\psi_0' + \left[\frac{\alpha_1(k)}{1+\alpha_1(k)} + \sigma\sigma'\frac{\beta_1(k)}{1+\beta_1(k)}\right]\psi_1\psi_1' \ , \tag{4.22}$$

where $\sigma\sigma' = +1$ or -1 for parallel and antiparallel spins, respectively. Note that the parameter $\alpha_0(k)$ is "shielded" by $[1+\alpha_0(k)]$ and hence it is reasonable to interpret α_0 as the finite k generalization of F_0^s.

Our result (4.22) for A is consistent with the Landau and weak coupling limits and retains the simplicity of both. It is possible to include dynamical shielding effects in A by introducing a frequency-dependent dielectric function. We reserve a discussion of these effects to future work.

V. LOW-TEMPERATURE SHEAR VISCOSITY

As one test of the phenomenological form of $\phi^{(c)}$ developed in Sec. IV we calculate the low-temperature shear viscosity η. We focus on this transport coefficient since its finite k and ω generalization is related to the linewidth of the zero-sound resonance in $S(k,\omega)$. We evaluate η by calculating the appropriate limit of the transverse momentum correlation function. Because of the inclusion of nonlocal effects in $\phi^{(c)}$, we obtain contributions to the shear viscosity which have no counterpart in calculations based on the local kinetic equation of Abrikosov and Khalatnikov.[19] We find that these nonlocal effects are not negligible: depending on the behavior of the functions $\alpha_i(k)$ and $\beta_i(k)$ for $k \gtrsim 2p_F$ they may increase the value of η to up to twice its local value.

Since the details of the calculation are lengthy, we present them in Appendix B. The formal analysis of the relationship between the shear viscosity η and the matrix elements of the memory function ϕ has been performed by a number of workers.[20,21] The relationship can be expressed as

$$\eta = \eta' + \eta'' \ , \tag{5.1}$$

where

$$\eta' = \lim_{k\to 0}\frac{-inm^*}{k^2}\langle 2|M(k,-i0^+)QM^{-1}(k,i0^+) \times QM(k,i0^+)|2\rangle \ , \tag{5.2}$$

$$\eta'' = \lim_{k\to 0}\frac{imm^*}{k^2}\langle 2|M(k,i0^+)|2\rangle \ , \tag{5.3}$$

where ψ_2 is proportional to p_x and the operator M is defined by

$$M_k(z) = M_k^{(0)} + M_k^{(s)} + M_k^{(c)}(z) \ , \tag{5.4}$$

with

$$\langle p|M_k^{(0)}|p'\rangle = \frac{\vec{k}\cdot\vec{p}}{m^*}\delta(p-p')F_k(p) \equiv \langle p|\omega_k|p'\rangle \ , \tag{5.5}$$

$$\langle p|M_k^{(s)}|p'\rangle = \frac{\vec{k}\cdot\vec{p}}{m^*}\phi^*(k,pp')F_k(p') \ , \tag{5.6}$$

$$\langle p|M_k^{(c)}(z)|p'\rangle = \phi^{(c)}(k,pp',z)F_k(p') \ . \tag{5.7}$$

The projection operator Q arises from the necessity of isolating the hydrodynamic singularities in the transverse current correlation function and is defined by

$$Q = 1 - \sum_{i=0}^{4}|i\rangle\langle i| \ . \tag{5.8}$$

The basis vectors $|i\rangle$ were introduced in Sec. III B. We recall that the first five $(i=0,\ldots,4)$ belong to the five hydrodynamic states. A factor of m^* rather than m appears in (5.2) and (5.3), since the state $|2\rangle$ refers to a quasiparticle transverse momentum.

At low temperatures, the "kinetic" term η' dominates the "direct" term η''. In the limit $T\to 0$ the local contribution to $\eta'(\eta_L)$ can be found by transforming (as done in Ref. 21) Eq. (5.2) into an integral equation which, if the momentum flux tensor is replaced by its local, free quasiparticle value, reduces to the exactly soluble[22] Abrikosov-Khalatnikov[19] transport equation. Since we wish to investigate the nonlocal effects, we adopt the alternative approach of using the method of kinetic modeling[23] in Eq. (5.2). That approach has been used extensively in calculations of $S(k,\omega)$, the physical quantity of main interest.

The matrix representation of the memory function is particularly useful for obtaining kinetic model[23] solutions of (5.2). We use the function space defined in Sec. III B to write

$$M(k,pp',z) = \sum_{ij}\psi_i(p)\psi_j(p')M_{ij}(k,z)F_k(p) \ , \tag{5.9}$$

where the matrix elements M_{ij} are defined as in (3.17). A kinetic model of order N is constructed by assuming that

$$M_{ij}=M_{ii}\delta_{ij} \tag{5.10}$$

for all i or $j > N$. The simplest kinetic model corresponds to the retention of only the first nonhydrodynamic matrix element M_{55} with the momentum state $\psi_5 \propto p_x p_z$. We substitute (5.10) with $N=5$ and (5.9) into (5.2) and obtain

$$\eta'_{1p}=\lim_{k\to 0}\frac{-imm^*}{k^2}M_{25}(k,i0^+)^2\,M_{55}(k,i0^+)^{-1} \ . \tag{5.11}$$

The "one-polynomial" solution η'_{1p} to (5.2) corresponds to the approximate Abrikosov and Khalatnikov[19] solution for η_L from the Landau theory. Since in the limit $T\to 0$, the latter approximation for η_L overestimates the exact solution[22] of the Landau theory by at most 25%, we might expect that η'_{1p} overestimates (5.2) by the same order.

In general there are three contributions to the matrix elements in (5.11) corresponding to the three terms (5.4) in M. However, as expected, only $M^{(c)}$ contributes to the nonhydrodynamic matrix element M_{55}. We show in Appendix B that in the limit $k\to 0$, $M_{55}^{(c)}$ can be written in the familiar form

$$M_{55}^{(c)}(k,i0^+)=i\tau^{-1} \ , \tag{5.12}$$

where the transverse relaxation time τ is real and independent of k. The local and nonlocal contributions to η' arise from $M_{25}^{(0)}$ and $M_{25}^{(c)}$, respectively. General symmetry arguments can be used to show that $M_{25}^{(s)}$ vanishes identically. At $T=0$, $M_{25}^{(0)}$ takes the form

$$M_{25}^{(0)}(k)=kp_F/m^*\sqrt{5} \ , \tag{5.13}$$

and in the limit $k\to 0$, $M_{25}^{(c)}$ can be written as (see Appendix B)

$$M_{25}^{(c)}(k,i0^+)=kp_F\Delta/m^*\sqrt{5} \ . \tag{5.14}$$

The dimensionless quantity Δ is real and independent of k. We combine (5.11)–(5.14) and write η'_{1p} in the more familiar form

$$\eta'_{1p}=\tfrac{1}{5}mm^*v_F^2(1+\Delta)^2\tau \ , \tag{5.15}$$

where $v_F=p_F/m^*$. Note that the form of (5.15) implies that Δ can be interpreted as a "renormalization" correction.

The evaluation of τ and Δ is discussed in Appendix B. We follow the method of Ref. 11 and express τ in the limit $T\to 0$ as a two-dimensional angular integral corresponding to quasiparticle collisions with momentum transfers $q\leqslant 2p_F$. The angular integrals are evaluated numerically using as input the calculated values of $\alpha_0(q)$ (see Fig. 1), assuming that α_1 β_0, and β_1 are independent of q for $q<2p_F$, and adopting the numerical values[24] $\alpha_1=F_1^s/3=2.01$, $\beta_0=F_0^a=-0.67$, and $\beta_1=F_1^a/3=-0.15$. As discussed in Sec. III B, these assumptions are consistent with those of Pines and co-workers.[4] The corresponding local contribution to η' is

$$\eta_L T^2=1.11\ \mathrm{P(mK)^2} \ . \tag{5.16}$$

Since $\alpha_0/(1+\alpha_0)\approx 1$ for $q\lesssim 2p_F$, η_L is insensitive to the magnitude of α_0; in contrast η_L depends sensitively on the values of $\beta_0(q)$ near $q=2p_F$.

At $T=0$ the form of the nonlocal contribution Δ to η' can be reduced to a one-dimensional integral over all q and involves the phenomenological parameters α_0, α_1, β_0, and β_1 and their derivatives. For $q\leqslant 2p_F$ we assume the same q dependence of these parameters as in the calculation of τ and obtain $\Delta_<=0.56$. Since the usual calculations of the low-temperature transport coefficients only involve momentum transfers $q<2p_F$, little is known about the q dependence of the parameters α_0, α_1, β_0 and β_1 for $q>2p_F$ except that they all approach zero for $q>>2p_F$. For this reason we can only estimate $\Delta_>$ and hence Δ with much uncertainty. The dependence of Δ on different assumed q dependences of the phenomenological parameters is discussed in Appendix B, where it is shown that reasonable bounds for the $q>2p_F$ contribution are $+0.02\gtrsim\Delta_>\gtrsim -0.38$. We adopt as an estimate of $\Delta_>$ the value $\Delta_>=-0.05$. This value corresponds to assuming that the magnitudes of α_0, β_0, and β_1 decrease linearly with increasing q and that all equal zero for $q>2.2p_F$. The latter value of q is obtained by linear extrapolation of the $\alpha_0(q)$ dependence shown in Fig. 1. The q dependence of α_1 is assumed to be negligible in the range $2p_F\leqslant q\leqslant 2.2p_F$. Our estimate for Δ with the above assumptions is $\Delta=\Delta_>+\Delta_<\simeq 0.5$ which yields the value $\eta T^2=2.5\ \mathrm{P(mK)^2}$. The lower bound to ηT^2, consistent with the smallest value of $\Delta_>$ is $\eta T^2=1.55\ \mathrm{P(mK)^2}$. The values are consistent with recent experimental measurements.[25] As indicated earlier, retention of additional matrix elements would likely reduce the magnitude of ηT^2 somewhat.

On the basis of the above result for η we conclude that our model for $\phi^{(c)}$ is at least qualitatively correct and that nonlocal corrections to the low-temperature transport coefficients cannot be neglected. There is currently some uncertainty in the experimental value of m^*/m and a value of $m^*/m=2.12$ has been proposed.[26] (But see also Ref. 27.) If we use the different[26] set of Landau parameters associated with the latter value of m^*/m we obtain a value of η which is lower than (5.16). In order to obtain agreement with experiment, the nonlocal contribution would have to be somewhat larger and hence the parameters $\alpha_l(q)$ and $\beta_l(q)$ would depend more strongly on q.

VI. DISCUSSION

We have seen that it is possible to develop a simple model for the dynamics of liquid ^{3}He based on a generalized kinetic theory approach. The relationships between this approach and the Landau[3] and polarization potential[4] theories were established and the mean-field nature of the latter was emphasized. The local and nonlocal contributions to the low-temperature shear viscosity of liquid ^{3}He were calculated using as input the values of the quasiparticle effective mass, the static structure function, and the spin-antisymmetric Landau parameters F_0^a and F_1^a. Reasonable agreement with experimental measurements of the shear viscosity was obtained, and the nonlocal contribution appears to be non-negligible. The main quantitative uncertainties in our present model arise from the unknown k dependences of our $k \neq 0$ generalizations of F_0^a and F_1^a.

Since the present calculation of the shear viscosity indicates that our simple form for the collisional part of the memory function is qualitatively correct, we plan to use the present formulation to calculate the dynamic structure function $S(k,\omega)$. This calculation[28] would involve a kinetic model solution of the generalized kinetic equation (2.7) similar to that performed for the viscosity except that matrix elements such as $M_{55}(k,z)$ must be evaluated at $k,z \neq 0$. Such a calculation would be of particular interest since the zero-sound mode in $S(k,\omega)$ as observed by neutron scattering[1,2] occurs at intermediate values of k and ω beyond the range of applicability of the Landau theory. Most of the presently available[4,29] calculations of neutron scattering from liquid ^{3}He are based on generalizations of the random phase approximation and hence the linewidth of zero sound in these calculations arises only from the decay of zero sound into single particle-hole excitations (Landau damping). However in the range of k, and T of experimental interest, neither this decay process[2] nor thermal broadening effects dominate the observed linewidth. The available experiments imply that the observed linewidth results from the decay of zero sound into multiparticle excitations.[2] Hence in the language of kinetic theory the observed zero-sound linewidth is associated with the nonlocal, k, and ω dependent behavior of the collisional part of the memory function due to incomplete quasiparticle collisions.

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APPENDIX A: DETERMINATION OF SPIN-SYMMETRIC PHENOMENOLOGICAL PARAMETERS

As discussed in Sec. III A, the phenomenological parameters $\alpha_0(k)$ and $\alpha_1(k)$ are determined from the sum rules (3.19) and (3.20) for $S(k,\omega)$. In this context the moments of $S(k,\omega)$ are obtained assuming that $\phi^{(c)}$ can be neglected and that ϕ^* is given by the two-parameter approximation (3.18). In order to obtain the moments of $S(k,\omega)$, we first solve for $S(k,pp',\omega)$ defined by [see (2.6)]

$$nS(k,pp',\omega) = \beta\omega \coth\frac{\beta\omega}{2}(1+e^{-\beta\omega})^{-1} \times \mathrm{Im}\,\mathcal{L}(k,pp',\omega+i0^+) \quad . \tag{A1}$$

We follow the same formal procedure as in Sec. III B, and obtain from (2.7) and (3.2) the result that for $\phi^{(c)}=0$

$$nS(k,pp',\omega) = \pi \coth\frac{\beta\omega}{2}(1+e^{-\beta\omega})^{-1} \times \langle p|\delta(\omega-\omega_k(1+\phi_k^*))F_k^0|p'\rangle \quad , \tag{A2}$$

where

$$\langle p|\omega_k|p'\rangle = \omega_k(p)\delta(p-p')F_k(p') \quad ,$$

$$F_k^0 = \omega_k(p)F_k(p) \quad ,$$

and

$$\omega_k(p) = \vec{k}\cdot\vec{p}/m^* \quad .$$

The evaluation of the first moment

$$S_1(k) = \int d^3p\, d^3p' \int \frac{d\omega}{2\pi}\omega \tanh\frac{\beta\omega}{2} S(k,pp',\omega) \tag{A3}$$

is straightforward. [We omit in (A3) and in the following all factors of 2 associated with the spin.] We obtain from (A2), (A3), and the matrix representation (3.16) of ϕ^* that at $T=0$

$$nS_1(k) = \frac{1}{2}\int d^3p\, \omega_k(p)^2 F_k(p) + \frac{1}{2}\sum_{ij}\phi_{ij}^*(k)\Lambda_i(k)\Lambda_j(k) \quad , \tag{A4}$$

where

$$\Lambda_i(k) = \int d^3p\, \omega_k(p)\psi_i(p)F_k(p) \quad . \tag{A5}$$

If we use the orthonormality condition (3.11) on ψ_i and the relation $\psi_1 \propto \omega_k(p)$ [see (3.14)], it is easy to show that (A4) reduces to (3.22) with $\alpha_1(k) = \phi_{11}^*(k)$. Note that (3.22) holds regardless of the number of matrix elements retained in ϕ^*.

The evaluation of the zeroth moment,

$$\tilde{S}(k,pp') = \int \frac{d\omega}{2\pi} S(k,pp'\omega) \ ,$$

involves the solution of a set of coupled equations. We use (A2) and the representation

$$\coth x = \frac{1}{x} + \frac{2x}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(x/\pi)^2 + n^2} \ , \tag{A6}$$

to write $\tilde{S}(k,pp')$ at $T=0$ in the from

$$\tilde{S}(k,pp') = 2\beta^{-1} \langle p \,|\, \omega_k (1+\phi_k^*) \,|\, \bar{p} \rangle \times \sum_{n=1}^{\infty} B(\bar{p},p',2\pi n/\beta) F_k^0(p') \ , \tag{A7}$$

where

$$B(pp',y) = \langle p \,|\, \{[\omega_k(1+\phi_k^*)]^2 + y^2\}^{-1} | p' \rangle \ . \tag{A8}$$

If we substitute the approximate form (3.18) for $\phi^*(k,pp')$ into (A7), it is straightforward to show that

$$\tilde{S}(k,pp') = \tilde{S}_0(k,pp') - 2\beta^{-1}\omega_k(p)^2 F_k^0(p') \sum_n B^*(pp',y_n)[\omega_k(p)^2 + y_n^2]^{-1} + 2\beta^{-1} F_k^0(p) F_k^0(p') \sum_n \sum_{i=0,1} \alpha_i \psi_i(p) B_i(p',y_n) \ , \tag{A9}$$

where $y_n = 2\pi n/\beta$. $\tilde{S}_0(k,pp')$ is the noninteracting quasiparticle limit of $\tilde{S}(k,pp')$ and is given by

$$\tilde{S}_0(k,pp') = 2\beta^{-1} F_k(p) \delta(p-p') \sum_n [\omega_k(p)^2 + y_n^2]^{-1} \ . \tag{A10}$$

The functions B^* and B_i and (A9) are defined as

$$B(pp',y) = \frac{\delta(p-p') - \omega_k(p) B^*(pp',y)}{\omega_k(p)^2 + y^2} \ , \tag{A11}$$

$$B_i(p',y) = \psi_i(\bar{p}) B(\bar{p}p',y) \ . \tag{A12}$$

If we also define

$$C_i(p',y) = \omega_k(\bar{p}) \psi_i(\bar{p}) B(\bar{p}p',y) \ , \tag{A13}$$

and substitute (3.18) into (A8), we find that B^* can be expressed as

$$B^*(pp',y) = \sum_{i=0,1} \alpha_i \psi_i(p) F_k(p) \left\{ C_i(p') + \sum_{j=0,1} [\omega_k(p)\delta_{ij} + \alpha_j \Lambda_{ij}] B_j(p') \right\} \ , \tag{A14}$$

where B_i satisfies the coupled equations

$$B_i(p) = \frac{\psi_i(p)}{\omega_k(p)^2 + y^2} - \sum_{j=0,1} \frac{\alpha_j \omega_k(\bar{p}) \psi_i(\bar{p}) \psi_j(\bar{p}) F_k(\bar{p})}{\omega_k(\bar{p})^2 + y^2} \left\{ C_j(p) + \sum_{e=0,1} [\omega_k(\bar{p})\delta_{je} + \alpha_e \Lambda_{je}] B_e(p) \right\} \ , \tag{A15}$$

with

$$\Lambda_{ij} = \omega_k(\bar{p}) \psi_i(\bar{p}) \psi_j(\bar{p}) F_k(\bar{p}) \ . \tag{A16}$$

C_i satisfies an equation similar to (A15) with $\psi_i(p)$ replaced by $\omega_k(p)\psi_i(p)$.

It is straightforward but tedious to solve the above coupled equations for B_0, B_1, C_0, and C_1 and to substitute these solutions into (A14) and (A9). The result for $\tilde{S}(k,pp')$ can be expressed as

$$\begin{aligned} \tilde{S}(k,pp') = {} & \tilde{S}_0(k,pp') + 2\beta^{-1} F_k^0(p) F_k^0(p') \\ & \times \sum_{n=0}^{\infty} \left\{ \frac{1}{1 + \alpha_0 I_0(y_n) + (1+\alpha_0)\alpha_1 I_1(y_n)} [\omega_k(p)^2 + y_n^2]^{-1} [\omega_k(p')^2 + y_n^2]^{-1} \right. \\ & \times \{ -\alpha_0 \psi_0^2 [\omega_k(p)\omega_k(p') - y_n^2] - \alpha_1 \psi_1(p)\psi_1(p')[\omega_k(p)\omega_k(p') - y_n^2] - \alpha_0\alpha_1 \omega_k(p) \\ & \left. \times [\omega_k(p) + \omega_k(p')] + \alpha_0 \alpha_1 \psi_0^2 [\omega_k(p)^2 + y_n^2][I_1(y_n) - \omega_k(p') I_0(y_n)] \} \right\} \ , \end{aligned} \tag{A17}$$

where

$$I_n(k,y) = [\omega_k(\bar{p})^2+y^2]^{-1}\omega_k(\bar{p})^2\,\psi_n(\bar{p})^2F_k(\bar{p}) \quad . \tag{A18}$$

If we integrate (A17) for $S(k,pp')$ over the momenta, take the limit $T\to 0$ and convert the sum over n into an integral, we find the result (3.21) for the structure function $S(k)$ in terms of $\alpha_0(k)$ and $\alpha_1(k)$.

APPENDIX B: EVALUATION OF MATRIX ELEMENTS

We present some of the techniques used to evaluate the matrix elements of $M^{(c)}$ discussed in Sec. V. Note that from (5.7), $M_{ij}^{(c)}$ is related to $\phi^{(c)}$ rather than the combination K [see (4.2)]. However, in the determination of $M_{2S}^{(c)}$ and $M_{SS}^{(c)}$, the simple form of $\tilde{L}$ given by (2.8), (3.3), and (3.18), implies that K can be replaced by $-\beta^{-1}\phi^{(c)}(kpp')F_k(p')$. All numerical factors associated with the spin are explicitly given. Thus for example (3.1) for $f(p)$ becomes

$$f(p) = \tfrac{1}{2}(2\pi)^{-3}(e^{\beta(p^2/2m^*-\mu)}+1)^{-1} \quad . \tag{B1}$$

$M_{SS}^{(c)}$ and hence τ can be found from (5.7), (5.12), (4.4), and (4.6). We have

$$\frac{1}{\tau} = \mathrm{Im}\left(\beta 2^2 \int d^3p\, d^3p'\, \psi_5(p)\psi_5(p')K(0,pp',i0^+)\right) \quad , \tag{B2a}$$

$$\begin{aligned}\tau^{-1} =& -2\pi N_S^2 2^5 \int d^3p_1\, d^3p_2\, d^3p_3\, d^3p_4 [\tfrac{1}{2} W_{xz}^{\uparrow\uparrow}(1234)^2 + W_{xz}^{\uparrow\downarrow}(1234)^2] \\ &\times \tfrac{1}{2}(2\pi)^3\delta(p_1+p_2-p_3-p_4)\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)(\epsilon_1+\epsilon_2-\epsilon_3=-\epsilon_4)^{-1} \\ &\times \tfrac{1}{2}[f(1)f(2)\tilde{f}(3)\tilde{f}(4)-\tilde{f}(1)\tilde{f}(2)f(3)f(4)] \quad ,\end{aligned} \tag{B2b}$$

where

$$W_{xz}(1234) = \int d^3p\, p_xp_z W(1234p) \quad , \quad \epsilon = p^2/2m^* \quad . \tag{B3}$$

The normalized form of ψ_5 at $T=0$ is

$$\psi_5(p) = N_5p_xp_z = [15/\nu(0)]^{1/2}p_xp_z/p_F^2 \quad . \tag{B4}$$

The factor of $\frac{1}{2}$ associated with the momentum δ function in (B2b) is a consequence of conservation of total spin of the interacting quasiparticles. The explicit form of W_{xz} is evaluated from (4.12) and can be written as

$$W_{xz}(1234)^2 = p_{1x}p_{1z}(p_{1x}p_{1z}+p_{2x}p_{2z}-p_{3x}p_{3z}-p_{4x}p_{4z})[A(p_1-p_3,p_1p_2)+A(p_1-p_3,p_3p_4)]^2 \quad . \tag{B5}$$

The energy δ function in (B2b) and the symmetry of the integrand imply that one can replace:

$$-\beta f(1)f(2)\tilde{f}(3)\tilde{f}(4)\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4) = \frac{f(1)f(2)\tilde{f}(3)\tilde{f}(4)-\tilde{f}(1)\tilde{f}(2)f(3)f(4)}{\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4} \quad . \tag{B6}$$

In order to avoid the factors of 2 and 2π associated with $f(p)$, we introduce the usual Fermi distribution function

$$n(p) = \{\exp[\beta(p^2/2m^*-\mu)]+1\}^{-1} \quad , \tag{B7a}$$

so that from (B1)

$$f(p) = \tfrac{1}{2}(2\pi)^{-3}n(p) \quad . \tag{B7b}$$

If we substitute (B5) and (B6) into (B2b) and use (B7) and $\tilde{n}(p) = 1-n(p)$, we obtain

$$\begin{aligned}\tau^{-1} =& \beta N_S^2(2\pi)^{-8}2\int d^3p_1\, d^3p_2\, d^3p_3\, d^3p_4\,(p_{1x}p_{1z})(p_{1x}p_{1z}+p_{2x}p_{2z}-p_{3x}p_{3z}-p_{4x}p_{4z}) \\ &\times \delta(p_1+p_2-p_3-p_4)\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)n(1)n(2)\tilde{n}(3)\tilde{n}(4) \\ &\times \tfrac{1}{4}\{\tfrac{1}{2}[A_{\uparrow\uparrow}(p_1-p_3,p_1p_2)+A_{\uparrow\uparrow}(p_1-p_3,p_3p_4)]^2+[A_{\uparrow\downarrow}(p_1-p_3,p_1p_2)+A_{\uparrow\downarrow}(p_1-p_3,p_3p_4)]^2\} \quad .\end{aligned} \tag{B8}$$

The right-hand side of (B8) can be evaluated in the $T\to 0$ limit by the method of Abrikosov and Khalatnikov.[19] We define the dimensionless energies

$$t = \beta(\epsilon_1-\mu), \quad x = \beta(\epsilon_3-\mu), \quad y = \beta(\epsilon_4-\mu) \quad , \tag{B9}$$

and the angles: θ and ϕ_2 are the polar and azimuthal angles of p_2 with respect to p; θ_1 and ϕ_1 are the same angles of p_1 with respect to k (the z axis); ϕ is the angle between the planes determined by p_1, p_2, and p_3, p_4. This choice of variables allows us after some algebra to write the $T\to 0$ limit of (B8) as

$$\tau^{-1}=2T^2N_5^2m^*p_F^5\nu(0)^{-2}(2\pi)^{-8}$$
$$\times\int_{-\mu}^{\infty}dt\int_{-\mu}^{\infty}dx\int_{-\mu}^{\infty}dy\; n(t)n(x+y-t)[1-n(x)][1-n(y)]$$
$$\times\int_0^{2\pi}d\phi\int_0^{\pi}\frac{d\theta\sin\theta}{2\cos\theta/2}\int_0^{\pi}d\theta_1\sin\phi_1\int_0^{2\pi}d\phi_2\int_0^{2\pi}d\phi_2\hat{p}_{1x}\hat{p}_{1z}$$
$$\times(\hat{p}_{1x}\hat{p}_{1z}+\hat{p}_{2x}\hat{p}_{2z}-\hat{p}_{3x}\hat{p}_{3z}-\hat{p}_{4x}\hat{p}_{4z})$$
$$\times\left[\tfrac{1}{2}A_{\uparrow\uparrow}^{2}\left[2\sin\frac{\theta}{2}\sin\frac{\phi}{2}p_F\right]+A_{\uparrow\downarrow}^{3}\left[2\sin\frac{\theta}{2}\sin\frac{\phi}{2}p_F\right]\right], \tag{B10}$$

where $n(x)=(e^x+1)^{-1}$, $\hat{p}$ is a unit vector and all factors of p_F and $\nu(0)$ are explicitly included. The form of $A_{\sigma\sigma'}(q)$ in (B10) is found from (3.13), (3.14), and (4.22) to be

$$A_{\sigma\sigma'}(q)=A_0^{\sigma\sigma'}(q)-\tfrac{1}{4}q^2A_1^{\sigma\sigma'}(q)\ , \tag{B11}$$

with

$$A_0^{\sigma\sigma'}(q)=L^{-1}(q/2p_F)\left[\frac{\alpha_0(q)}{1+\alpha_0(q)}+\sigma\sigma'\frac{\beta_0(q)}{1+\beta_0(q)}\right] \tag{B12}$$

and

$$A_1^{\sigma\sigma'}(q)=3\left[\frac{\alpha_1(q)}{1+\alpha_1(q)}+\sigma\sigma'\frac{\beta_1(q)}{1+\beta_1(q)}\right]\ . \tag{B13}$$

The limits of integration over the energy variables t, x, and y can be extended to $-\infty$, and the energy integrals in (B10) can be done exactly[30] to yield $\frac{2}{3}\pi^2$. The angular integration over ϕ_1 can be performed using an identity due to Sykes and Brooker[22] with the result

$$\int_0^{\pi}d\theta_1\sin\theta_1\int_0^{2\pi}d\phi_1\int_0^{2\pi}d\phi_2\,\hat{p}_{1x}\hat{p}_{1z}(\hat{p}_{1x}\hat{p}_{1z}+\hat{p}_{2x}\hat{p}_{2z}-\hat{p}_{3x}\hat{p}_{3z}-\hat{p}_{4x}\hat{p}_{4z})=\tfrac{32}{5}\pi^2\sin^4\frac{\theta}{2}\sin^2\frac{\phi}{2}\cos^2\frac{\phi}{2}\ . \tag{B14}$$

We use the above results and let $\theta/2\to\theta$, $\phi/2\to\phi$ to write (B10) as

$$\tau^{-1}=\frac{4\pi^2T^2m^*}{p_F^2}(\tfrac{1}{2}\lambda_{\uparrow\uparrow}+\lambda_{\uparrow\downarrow})\ , \tag{B15}$$

where

$$\lambda_{\sigma\sigma'}=\int_0^{\pi/2}d\theta\sin^5\theta\int_0^{\pi/2}d\phi\sin^2\phi\cos^2\phi A_{\sigma\sigma'}^{2}(2\sin\theta\sin\phi p_F)\ . \tag{B16}$$

Then, using (5.16) and (B15) and inserting the appropriate factors of $\hbar$ and k_B, we find that the local contribution to the viscosity can be expressed as

$$\eta_LT^2=\tfrac{1}{5}nm^*v_F^2\frac{\hbar T_F}{k_B}(2\pi^2\lambda)^{-1} \tag{B17}$$

where $\lambda=\frac{1}{2}\lambda_{\uparrow\uparrow}+\lambda_{\uparrow\downarrow}$. The angular integrals in (B16) are evaluated numerically using the values of the parameters α_0, β_0, α_1, and β_1 discussed in Sec. V. We find the result $\lambda=0.204+0.645=0.849$ from which is found the numerical value given in (5.17) for η_L.

In the same manner the nonlocal matrix element $M_{25}^{(c)}$ has the form

$$M_{25}^{(c)}(k,z)=2^2N_2N_5\int d^3p_1\,d^3p_2\,d^3p_3\,d^3p_4\,[\tfrac{1}{2}W_x^{\uparrow\uparrow}(1234)W_{xz}^{\uparrow\uparrow}(1234)+W_x^{\uparrow\downarrow}(1234)W_{xz}^{\uparrow\downarrow}(1234)]$$
$$\times[G(1234,kz)-G(1234,-k-z)]\ , \tag{B18}$$

where

$$W_x = \int d^3p\, p_x W(1234p) \tag{B19}$$

and $N_2 = [3/\nu(0)]^{1/2} p_F^{-1}$. In order to extract the leading k dependence of $M_{25}^{(\epsilon)}$ we write [see (4.6)–(4.8)]

$$G(1234, \pm k \pm z) = \tfrac{1}{2}(2\pi)^3 \delta_\pm(1234) H_\pm(1234)[\pm z - E_\pm(1234)]^{-1} , \tag{B20}$$

where

$$\delta_\pm(1234) = \delta(p_1 + p_2 - p_3 - p_4 \pm k) = \delta_0 \pm k\delta_1 + \cdots , \tag{B21}$$

$$H_\pm(1234) = B(1234, \pm k) E^{-1}(1234, \pm k) = H_0 \pm k H_1 + \cdots , \tag{B22}$$

and

$$E_\pm^{-1}(1234) = E(1234, \pm k)^{-1} = E_0^{-1} \pm k E_1^{-1} + \cdots . \tag{B23}$$

We substitute (B20)–(B23) in (B18) and obtain to $O(k)$

$$M_{25}^{(\epsilon)}(k, i0^+) = -2N_2N_5(2\pi)^3 \int d^3p_1\, d^3p_2\, d^3p_3\, d^3p_4 [\tfrac{1}{2} W_x^{\downarrow\uparrow\uparrow}(1234) W_{xz}^{\downarrow\downarrow\uparrow}(1234) + W_x^{\downarrow\uparrow\downarrow}(1234) W_{xz}^{\downarrow\downarrow\downarrow}(1234)]$$
$$\times [2\pi i \delta_0 H_0 \delta(E_0) + 2k(\delta_1 H_0 E_0^{-1} + \delta_0 H_1 E_0^{-1} + \delta_0 H_0 E_1^{-1})] . \tag{B24}$$

The form (4.12) of W implies that $W_x\delta_0 = 0$, and hence only the term proportional to δ_1 in (B24) is nonzero. If we substitute the explicit forms of H_0, E_0, N_2, and N_5, write $W_x\delta_1 = -W_x^z\delta_0$ with $W_x^z = dW_x/dp_{1z}$, let $\vec{q} = \vec{p}_3 - \vec{p}_1 = \vec{p}_2 - \vec{p}_4$, and use the definition (5.14) of Δ, we can express (B24) as

$$\Delta = -15\pi^6 \int \frac{d^3q\, d^3p_1\, d^3p_2}{(2\pi)^9} [\tfrac{1}{2} W_x^{z\uparrow\uparrow}(q,p_1p_2) W_{xz}^{\downarrow\uparrow}(q,p_1p_2) + W_x^{z\uparrow\downarrow}(q,p_1p_2) W_{xz}^{\downarrow\downarrow}(q,p_1p_2)]$$
$$\times [n(p_1)n(p_2)\tilde{n}(p_1+q)\tilde{n}(p_2-q) - \tilde{n}(p_1)\tilde{n}(p_2)n(p_1+q)n(p_2-q)][\vec{q}\cdot(\vec{p}_2 - \vec{p}_1) - q^2]^{-2} \tag{B25}$$

with all momenta in units of p_F. $W_x^z(q,p_1p_2)$ and $W_{xz}(q,p_1p_2)$ can be written as

$$W_x^z(\vec{q}, \vec{p}_1\vec{p}_2) = \frac{q_xq_z}{q}\left[\frac{dA_0(q)}{dq} + \alpha(\vec{q}, \vec{p}_1\vec{p}_2)\frac{dA_1(q)}{dq}\right]$$
$$+ A_1(q)\left[\frac{\vec{q}\cdot\vec{p}_1}{q^2} q_x \left(p_2 + \frac{q}{2}\right)_z + \frac{\vec{q}\cdot\vec{p}_2}{q^2} q_x \left(p_1 - \frac{q}{2}\right)_z - \frac{2q_xq_z}{q^2}(\vec{q}\cdot\vec{p}_1)(\vec{q}\cdot\vec{p}_2) + q_x\frac{(p_2-p_1)_z}{2}\right] , \tag{B26}$$

$$W_{xz}(\vec{q}, \vec{p}_1\vec{p}_2) = [q_x(p_2-p_1)_z + q_z(p_2-p_1)_x - 2q_xq_z][A_0(q) + A_1(q)A(\vec{q}, \vec{p}_1\vec{p}_2)] , \tag{B27}$$

where

$$\alpha(\vec{q}, \vec{p}_1\vec{p}_2) = \frac{1}{2q^2}[q^2(\vec{q}\cdot\vec{p}_2 - \vec{q}\cdot\vec{p}_1) + 2(\vec{q} - \vec{p}_1)(\vec{q} - \vec{p}_2) - q^4] . \tag{B28}$$

The functions $A_0(q)$ and $A_1(q)$ in (B26) and (B27) are defined in (B11) and (B12), respectively. The two products of statistical factors in (B25) can be combined by noting that $W_x \to W_x$ and $W_{xz} \to W_{xz}$ under the change of variables $\vec{p}_1 \to \vec{p}_1 + \vec{q}$, $\vec{p}_2 \to \vec{p}_2 - \vec{q}$, and $\vec{q} \to -\vec{q}$. The integration over the momenta in (B25) can be facilitated by introducing the additional variable $\nu = \vec{q}\cdot\vec{p}_1 + q^2/2$ and writing Δ as

$$\Delta = -30\pi^6 \int_{-\infty}^{\infty} d\nu \int \frac{d^3q}{(2\pi)^3} [\tfrac{1}{2} T_{\uparrow\uparrow}(q,\nu) + T_{\uparrow\downarrow}(q,\nu)] , \tag{B29}$$

where

$$T(q,\nu) = \int \frac{d^3p}{(2\pi)^3} n(p)\tilde{n}(p+q)\delta(\nu - \vec{q}\cdot\vec{p} - q^2/2) \int \frac{d^3p'}{(2\pi)^3} \frac{n(p')\tilde{n}(p'-q)\, W_x^z(q,pp')\, W_{xz}(q,pp')}{(\vec{q}\cdot\vec{p} - q^2/2 - \nu)^2} . \tag{B30}$$

If we substitute (B26)–(B28) into (B30), $T(q,\nu)$ can be expressed in terms of the functions

$$R_{nm}^{x,z}=\int\frac{d^3p}{(2\pi)^3}\frac{(\hat{q}\cdot\vec{p})^n p_{x,z}^m n(p)\tilde{n}(p-q)}{(\vec{q}\cdot\vec{p}-q^2/2-\nu)^2}\;, \tag{B31}$$

$$P_{nm}^{x,z}=\int\frac{d^3p}{(2\pi)^3}(\hat{q}\cdot\vec{p})^n p_{x,z}^m n(p)\tilde{n}(p+q)\delta(\nu-\vec{q}\cdot\vec{p}-q^2/2)\;. \tag{B32}$$

We write $R_n=R_{n,o}^{x,z}$, $P_n=P_{n,o}^{x,z}$, use the identity $R_{n,m}^{x}=(q_x/q)^m R_{n+m}$ and analogous identities for R_{nm}^{z} and P_{nm}^{z}, and find after much algebra that

$$T(q,\nu)=\frac{2q_x^2q_z^2}{q^2}\left(\frac{dA_0(q)}{dq}+\frac{dA_1(q)}{dq}\right)[A_0(q)T_0(q,\nu)+A_1(q)T_1(q,\nu)]\;, \tag{B33}$$

where

$$T_0(q,\nu)=P_0(q,\nu)\tilde{R}_1(q,\nu)\;, \tag{B34}$$

$$T_1(q,\nu)=(\nu^2/q^2-q^2/4)P_0(q,\nu)\tilde{R}_1(q,\nu)+(\nu/q)P_0(q,\nu)\tilde{R}_2(q,\nu)\;. \tag{B35}$$

The functions $\tilde{R}_1$ and $\tilde{R}_2$ are defined as

$$\tilde{R}_1(q,\nu)=R_1(q,\nu)-(\nu/q+q/2)R_0(q,\nu)\;, \tag{B36}$$

$$\tilde{R}_2(q,\nu)=R_2(q,\nu)-2(\nu/q+q/2)R_1(q,\nu)+(\nu/q+q/2)^2R_0(q,\nu)\;. \tag{B37}$$

P_0 can be evaluated analytically at $T=0$ using either geometrical arguments or a general method due to duBois[31] to satisfy the constraints $p<1$, $|p+q|>1$, and $\vec{p}\cdot\vec{q}+q^2/2=\nu$. The result is (note that all momenta are in units of p_F)

$$P_0(q,\nu)=\begin{cases}(8\pi^2q)^{-1}[1-(\nu/q-\frac{1}{2}q)^2], & \frac{1}{2}q^2+q\geq\nu\geq\frac{1}{2}q^2-q;\; q\geq 2\\ (8\pi^2q)^{-1}[1-(\nu/q-\frac{1}{2}q)^2], & q+\frac{1}{2}q^2\geq\nu\geq q-\frac{1}{2}q^2;\; q\leq 2\\ 2\nu(8\pi^2q)^{-1}, & 0\leq\nu\leq q-q^2/2;\; q\leq 2\;.\end{cases} \tag{B38}$$

The evaluation of $\tilde{R}_1$ and $\tilde{R}_2$ at $T=0$ is also straightforward. We have

$$\tilde{R}_1(q\geq 2)=-(4\pi^2q^2)^{-1}\{(\nu/q+\tfrac{1}{2}q)+\tfrac{1}{2}[1-(\nu/q+\tfrac{1}{2}q)^2]\ln[(\nu/q+\tfrac{1}{2}q+1)/(\nu/q+\tfrac{1}{2}q-1)]\}\;, \tag{B39a}$$

$$\tilde{R}_1(q\leq 2)=-(4\pi^2q^2)^{-1}\{\tfrac{1}{2}(\nu+q)+\nu\ln(\nu/q)+\tfrac{1}{2}[1-(\nu/q+\tfrac{1}{2}q)^2]\ln(\nu/q+\tfrac{1}{2}q+1)$$
$$-\tfrac{1}{2}[1-(\nu/q-\tfrac{1}{2}q)^2]\ln(\nu/q-\tfrac{1}{2}q+1)\}\;, \tag{B39b}$$

$$\tilde{R}_2(q\geq 2)=(4\pi^2q^2)^{-1}\tfrac{1}{2}q(1-q^2/12)\;, \tag{B40a}$$

$$\tilde{R}_2(q\leq 2)=(4\pi^2q^2)^{-1}\tfrac{2}{3}\;. \tag{B40b}$$

The simple ν dependence of T_0 and T_1 given above allows us to perform the ν integration in (B29) analytically. We find after tedious algebra that the matrix element Δ can be reduced to

$$\Delta=\tfrac{1}{32}[\tfrac{1}{2}(\Delta_0^{\uparrow\uparrow}+\Delta_1^{\uparrow\uparrow})+(\Delta_0^{\uparrow\downarrow}+\Delta_1^{\uparrow\downarrow})]\;, \tag{B41}$$

where

$$\Delta_0^{\sigma\sigma'}=\int_0^\infty dq\, qJ_0(q)A_0^{\sigma\sigma'}(q)\frac{d}{dq}[A_0^{\sigma\sigma'}(q)+A_1^{\sigma\sigma'}(q)]\;, \tag{B42}$$

$$\Delta_1^{\sigma\sigma'}=\int_0^\infty dq\, qJ_1(q)A_1^{\sigma\sigma'}(q)\frac{d}{dq}[A_0^{\sigma\sigma'}(q)+A_1^{\sigma\sigma'}(q)]\;. \tag{B43}$$

The functions J_0 and J_1 can be expressed as

$$J_0(q\geq 2)=\tfrac{1}{15}q^2(22+q^2)+q(\tfrac{8}{15}-\tfrac{2}{3}q^2-\tfrac{1}{3}q^3+\tfrac{1}{60}q^5)\ln(1+2/q)-q(\tfrac{8}{15}-\tfrac{2}{3}q^2+\tfrac{1}{3}q^3-\tfrac{1}{60}q^5)\ln(1-2/q)\;, \tag{B44a}$$

$$J_0(q \leq 2) = q^3(\tfrac{2}{30}q - \tfrac{1}{40}q^2) - \tfrac{4}{3}q^3\ln 2 + q(\tfrac{8}{15} + \tfrac{1}{2}q - \tfrac{1}{12}q^3 + \tfrac{1}{160}q^5)\ln(1+\tfrac{1}{2}q)$$
$$+ q(\tfrac{8}{15} - \tfrac{1}{2}q + \tfrac{1}{12}q^3 - \tfrac{1}{160}q^5)\ln(1-q/2) \;, \tag{B44b}$$

$$J_1(q \geq 2) = \tfrac{46}{315}q^2 + \tfrac{6}{35}q^4 - \tfrac{1}{70}q^6 + q(\tfrac{8}{105} - \tfrac{2}{5}q^2 - \tfrac{1}{3}q^3 + \tfrac{1}{20}q^5 - \tfrac{1}{280}q^7)\ln(1+2/q)$$
$$- q(\tfrac{8}{105} - \tfrac{2}{5}q^2 + \tfrac{1}{3}q^3 - \tfrac{1}{20}q^5 + \tfrac{1}{280}q^7)\ln(1-2/q) \;, \tag{B45a}$$

$$J_1(q \leq 2) = \tfrac{53}{210}q^3 - \tfrac{29}{2520}q^5 + \tfrac{1}{168}q^7 - \tfrac{4}{5}q^3\ln 2 + q(\tfrac{8}{105} + \tfrac{1}{6}q - \tfrac{1}{12}q^3 + \tfrac{3}{160}q^5 - \tfrac{1}{672}q^7)\ln(1+\tfrac{1}{2}q)$$
$$+ q(\tfrac{8}{105} - \tfrac{1}{6}q + \tfrac{1}{12}q^3 - \tfrac{3}{160}q^5 + \tfrac{1}{672}q^7)\ln(1-2/q) \;. \tag{B45b}$$

Note that J_0 and J_1 and their first derivatives are continuous at $q=2$.

The one-dimensional integrals in (B42) and (B43) are performed numerically using as input the phenomenological parameters α_0, α_1, β_0, and β_1 and their derivatives. The known q dependence of α_0 and the weak q dependence of α_1, β_0 and β_1 for $q \leq 2$ allows us to estimate the contribution to Δ with some degree of confidence. We find that the contribution to Δ from $q \leq 2$ is $\Delta_< \approx 0.56$ with the q dependence of the phenomenological parameters assumed to be the same as in the calculation of τ. For $q > 2$ we determine the q dependence of α_0 by linear extrapolation of its behavior near $q=2$; we set $\alpha_0 = 0$ for $q > 2.2$. The q dependence of α_1, β_0 and β_1 for $q > 2$ is not known. In order to estimate the contribution to Δ from $q > 2$, we determine the sensitivity of Δ, to variations in these parameters. (Note that $\Delta_> = 0$ if all the phenomenological parameters were assumed to be independent of q.) The simplest assumption is that α_1, β_0 and β_1 are relatively constant for $2 > q > 2.2$ so that the only contribution to $\Delta_>$ arises from the decrease in α_0 in this range. The result is $\Delta_> = +0.02$. More realistic assumptions for the q dependence of β_0 and β_1 cause $\Delta_>$ to be negative. For example, if we assume that β_0 and β_1 decrease linearly (in magnitude) with constant slope δ, we find that $\Delta_>$ depends weakly on δ and that $\Delta_> \approx -0.3$. Another assumption is that β_0 and β_1 decrease linearly with slopes δ_0 and δ_1. If we determine δ_0 and δ_1 by requiring that both β_0 and β_1 become zero for $q=2.2$, we obtain $\Delta_> = -0.05$.

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